

A HIGHER BACHMANN-HOWARD PRINCIPLE

ANTON FREUND

ABSTRACT. We present a higher well-ordering principle which is equivalent (over Simpson’s set theoretic version of **ATR**₀) to the existence of transitive models of Kripke-Platek set theory, and thus to Π_1^1 -comprehension. This is a partial solution to a conjecture of Montalbán and Rathjen: partial in the sense that our well-ordering principle is less constructive than demanded in the conjecture.

CONTENTS

1. Search Trees for Admissible Sets	4
2. A Higher Bachmann-Howard Construction	11
3. From Search Tree to Proof Tree	24
4. Cut Elimination	28
5. Collapsing Proofs	41
6. A Well-Ordering Proof	59
References	64

The present work is situated at the intersection of set theory, proof theory and reverse mathematics. Accordingly, we motivate our result from two different viewpoints: Beginning with the set theoretic perspective, we invite the reader to recall axiom beta (see e.g. [Bar75, Definition I.9.5]): It states that any well-founded relation can be collapsed to the \in -relation. In particular this turns well-foundedness into a Δ -notion. Not surprisingly, then, axiom beta adds considerable strength to theories which admit Δ -separation and Σ -collection, such as Kripke-Platek set theory: We can now form the set of arithmetical well-orderings on the natural numbers. This implies the existence of the Church-Kleene ordinal ω_1^{CK} and of the set $\mathbb{L}_{\omega_1^{\text{CK}}}$, which is a transitive model of Kripke-Platek set theory with infinity (see [Bar75, Corollary V.5.11]; such models will henceforth be called admissible sets). Also, the combination of axiom beta and Σ -collection is known to imply Δ_2^1 -comprehension for subsets of the natural numbers (see [Poh98, Theorem 3.3.4.7]). For weaker set theories, which do not contain Δ -separation, the contribution of axiom beta can be quite different: Consider for example the set-theoretic version of **ATR**₀ introduced by Simpson (see [Sim09, Section VII.3]). This set theory contains axiom beta but does not prove Π_1^1 -comprehension or the existence of admissible sets. From a set theoretic perspective, the higher well-ordering principle that we shall introduce can be seen as a strengthening of axiom beta: one that implies the existence of admissible sets, even when the base theory does not contain Δ -separation.

From a different viewpoint, the present paper is part of an ongoing investigation into the reverse mathematics of well-ordering principles. A typical result in this area (due to Marcone and Montalbán [MM11], and re-proved by Afshari and Rathjen [AR09] using different methods) says that the following statements are equivalent over \mathbf{RCA}_0 :

- (i) the ω -th jump of any set exists;
- (ii) for any well-ordering X , a certain term system ε_X is well-ordered as well;
- (iii) any set is contained in a countable coded ω -model of \mathbf{ACA} .

A function such as $X \mapsto \varepsilon_X$ in (ii), which maps any well-ordering to some other well-ordering, is called a well-ordering principle. Many other set existence axioms have been characterized in terms well-ordering principles: arithmetical comprehension (Girard [Gir87, Theorem 5.4.1], Hirst [Hir94]), Π^0_{ω} -comprehension (Marcone and Montalbán [MM11]), arithmetical transfinite recursion (Friedman, Montalbán and Weiermann [FMW], Rathjen and Weiermann [RW11], Marcone and Montalbán [MM11]), the existence of ω -models of arithmetical transfinite recursion (Rathjen [Rat14]), and the existence of ω -models of bar induction (Rathjen and Vizcaíno [RV15]). While these results have reached considerable proof theoretic strength they share a limitation in terms of logical complexity: Statements analogous to (ii) and (iii) are equivalent to Π^1_2 -formulas. Thus we cannot expect to replace (i) by a genuine Π^1_3 -statement, such as the axiom of Π^1_1 -comprehension. To overcome this limitation Rathjen [Rat11, Rat14] and Montalbán [Mon11, Section 4.5] have proposed to use well-ordering principles of higher type, i.e. functionals \mathcal{F} which map each well-ordering principle f to a well-ordering $\mathcal{F}(f)$. At the place of (ii) above one would state that

- (\star) “for any well-ordering principle f , the set $\mathcal{F}(f)$ is a well-ordering”.

Alternatively, one could consider functionals which map well-ordering principles to well-ordering principles (i.e. increasing the type of the co-domain as well). At the place of (iii) one demands the existence of β -models. Note that the resulting statements are Π^1_3 -formulas. By [Sim09, Theorem VII.2.10] the existence of countable coded β -models is equivalent to Π^1_1 -comprehension, as desired. A strategy to construct β -models from higher well-ordering principles has also been suggested by Rathjen (personal communication; cf. also [Rat11] and [Rat14, Section 6]): In the results cited above, Rathjen and collaborators construct the required ω -models via Schütte’s method of search trees (or “deduction chains”, see [Sch77, Section II.3]). In order to construct β -models, Rathjen proposed to extend these ideas to Girard’s [Gir85, Section 6] notion of β -proof. This fits well with Montalbán’s [Mon11] remark that the variable f in (\star) should range over dilators (note that the quantification needs to be restricted in some way if we are to remain within the realm of second order arithmetic). In the present paper we do not work with β -proofs and dilators in the strict sense, but the underlying ideas are still of central importance (for more on this point see Remark 1.8 below). Let us point out that we will construct transitive models of Kripke-Platek set theory rather than β -models, thus realizing a modification of Rathjen’s idea. As Kripke-Platek set theory does not prove axiom beta this does not immediately yield β -models of second order arithmetic (in contrast, the theory $\mathbf{ATR}_0^{\text{set}}$ in [Sim09, Theorem VII.3.27] contains axiom beta). Nevertheless, the existence of transitive models of Kripke-Platek set theory is known to imply Π^1_1 -comprehension (see [Poh98, Theorem 3.3.3.5]).

It is time to describe our higher well-ordering principle in detail. The meta-theory of the present paper will be primitive recursive set theory with infinity (see e.g. [Rat92b, Section 6]). Note that “primitive recursive” will always mean “primitive recursive relative to ω ”. In the language of this theory we have function symbols for all primitive recursive (class) functions. We cannot quantify over all of these functions, but we can use Skolemization to quantify over parametrized families: Given a primitive recursive function $(u, x) \mapsto F(u, x)$ quantification over the functions $x \mapsto F(u, x)$ amounts to first-order quantification over the set parameter u , which is of course permitted. In particular we can implement well-ordering principles as follows (cf. Definition 2.1 below): Consider primitive recursive functions $T : (u, \alpha) \mapsto (T_\alpha^u, <_{T_\alpha^u})$ and $|\cdot|_T : (u, s) \mapsto |s|_T^u$. We say that T^u is a (ranked compatible) well-ordering principle, abbreviated as $\text{WOP}(T^u)$, if the following holds:

- (WOP1) For every ordinal α the set T_α^u is well-ordered by $<_{T_\alpha^u}$.
- (WOP2) For $\alpha < \beta$ we have $T_\alpha^u \subseteq T_\beta^u$ and $<_{T_\alpha^u} = <_{T_\beta^u} \cap (T_\alpha^u \times T_\alpha^u)$; furthermore we have $T_\lambda^u = \bigcup_{\alpha < \lambda} T_\alpha^u$ for each limit ordinal λ .
- (WOP3) We have

$$|s|_T^u = \begin{cases} \min\{\alpha \in \text{Ord} \mid s \in T_{\alpha+1}^u\} & \text{if such an } \alpha \text{ exists,} \\ \{1\} & \text{otherwise.} \end{cases}$$

As the set $\{1\}$ is not an ordinal (which is its sole purpose) this does, in particular, make the class $T^u = \bigcup_{\alpha \in \text{Ord}} T_\alpha^u$ primitive recursive. Considering the general form (\star) of a higher well-ordering principle, we should now describe a functional \mathcal{F} which transforms each well-ordering principle T^u into a well-ordering $\mathcal{F}(T^u)$. Viewing the set u rather than the class function T^u as the argument we should strive to define \mathcal{F} as a primitive recursive class function. In the present paper we take a more abstract approach: Rather than constructing a term system $\mathcal{F}(T^u)$ explicitly we axiomatize ordinals of the desired order-type. The following notion achieves this (cf. Definition 2.2 below): Given a well-ordering principle T^u and an ordinal α , a function $\vartheta : T_\alpha^u \rightarrow \alpha$ is called a Bachmann-Howard collapse, abbreviated as $\vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$, if the following holds for all $s, t \in T_\alpha^u$:

- (BH1) $|s|_T^u < \vartheta(s)$;
- (BH2) if $s <_{T_\alpha^u} t$ and $|s|_T^u < \vartheta(t)$ then $\vartheta(s) < \vartheta(t)$.

These conditions are motivated by Rathjen’s notation system for the Bachmann-Howard ordinal (see e.g. [RV15]). Now we can define the central notion of the present paper: The higher Bachmann-Howard principle for T is the statement

$$\text{BH}(T) := \forall u (\text{WOP}(T^u) \rightarrow \exists \alpha \exists \vartheta \vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha).$$

From a set theoretic perspective one may view this as a strengthening of axiom beta (for linear orderings): Rather than collapsing the single well-ordering T_0^u we demand a “compatible collapse of a compatible family”. The reader may wish to consider the simple Example 2.5, the existence proof in Remark 2.6, and the connection with axiom beta established in Remark 2.7.

Let us now state our main results. In Section 6 we will show that primitive recursive set theory proves the following, for each primitive recursive function T : If $\text{WOP}(T^u)$ holds and \mathbb{A} is an admissible set with $u \in \mathbb{A}$ then there is a Bachmann-Howard collapse $\vartheta : T_{o(\mathbb{A})}^u \xrightarrow{\text{BH}} o(\mathbb{A})$, where $o(\mathbb{A}) = \mathbb{A} \cap \text{Ord}$. The other sections are devoted to the converse direction: We will define a primitive recursive function

which maps each countable transitive set $u = \{u_i \mid i \in \omega\}$ (with fixed enumeration) and each ordinal α to a linear ordering $\varepsilon(S_{\omega^\alpha}^u)$, such that the following holds:

Theorem. *Working in primitive recursive set theory, consider a countable transitive set u . If the implication*

$$\text{WOP}(\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)) \rightarrow \exists_\alpha \exists_\vartheta \vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$$

holds then there is an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$.

If u is hereditarily countable then we may apply the theorem to the transitive closure of $\{u\}$, to get an admissible set \mathbb{A} with $u \in \mathbb{A}$. This completes the equivalence between (ii) and (iii) of the following theorem. The equivalence between (i) and (iii) is known (see [Poh98, Theorem 3.3.3.5] and [Jäg86, Lemma 7.5], and use [Sim09, Section VII.3] to relate second order arithmetic and set theory).

Theorem. *The following axioms resp. axiom schemes are equivalent over primitive recursive set theory, extended by the axiom of countability and axiom beta:*

- (i) Π_1^1 -comprehension for subsets of the natural numbers;
- (ii) the higher Bachmann-Howard principle, i.e. the collection of axioms $\text{BH}(T)$ for all primitive recursive functions T ;
- (iii) the statement that every set is an element of an admissible set.

Let us make the connection with second order arithmetic: Eliminating the primitive recursive function symbols, it should be straightforward to show that the base theory of the theorem is conservative over Simpson's [Sim09, Section VII.3] set theoretic version of \mathbf{ATR}_0 . Thus the equivalence between (i), (ii) and (iii) translates into a theorem of \mathbf{ATR}_0 . Note that (ii) and (iii) are Π_2 -statements in the language of set theory. According to [Sim09, Theorem VII.3.24] they correspond to Π_3^1 -statements of second order arithmetic, as expected. In a sense, we have thus solved the conjecture formulated by Montalbán [Mon11, Section 4.5] and Rathjen [Rat14, Section 6]. To see why our solution is not completely satisfying, let us once again compare our Bachmann-Howard principles $\text{BH}(T)$ with the general form (\star) of a well-ordering principle: The point is that $\text{BH}(T)$ merely asserts the existence of ordinals with certain properties, while (\star) requires to compute these ordinals (or some well-orderings of the appropriate order type) by a concrete functional \mathcal{F} . The author currently works on a more constructive version of the present paper, including an explicit definition of \mathcal{F} .

1. SEARCH TREES FOR ADMISSIBLE SETS

In this section we give a primitive recursive construction $(u, \alpha) \mapsto S_\alpha^u$ of “search trees” for each ordinal α and each countable transitive set u . To be more precise, S_α^u will be primitive recursive in α and a given enumeration $u = \{u_i \mid i \in \omega\}$ of u ; for the sake of readability this enumeration will often be left implicit. One may think of S_α^u as an attempted proof of a contradiction in \mathbb{L}_α^u -logic, with the axioms of Kripke-Platek set theory as open assumptions. Recall that the distinctive rule of \mathbb{L}_α^u -logic allows to infer $\forall_x \varphi(x)$ from the assumptions $\varphi(a)$ for all $a \in \mathbb{L}_\alpha^u$. If \mathbb{L}_α^u satisfies the Kripke-Platek axioms then the construction of S_α^u cannot be successful, by the correctness of \mathbb{L}_α^u -logic; this will manifest itself in the fact that S_α^u turns out ill-founded. Search trees are distinguished by a converse property: We will be able to transform an infinite branch of S_α^u into a standard model $u \subseteq M \subseteq \mathbb{L}_\alpha^u$ of the

Kripke-Platek axioms. The method of search trees (or “deduction chains”) is due to Schütte, who used it to prove the completeness theorem for first-order logic (see [Sch77, Section II.3]). The present paper is mainly influenced by Rathjen’s use of search trees in the construction of ω -models (cf. the introduction). Another application of search trees in ω -logic is due to Jäger and Strahm [JS99]. The author knows of one application of “ β -search trees”: This is Buchholz’ [Buc88] construction of a dilator that bounds the stages of inductive definitions.

Recall that the constructible hierarchy relative to u can be given as a primitive recursive function $(u, \alpha) \mapsto \mathbb{L}_\alpha^u$ (see [Rat92b, Definition 2.3]). Let us emphasize that the enumeration $u = \{u_i \mid i \in \omega\}$ does *not* come into play at this point: The stage \mathbb{L}_0^u is simply the set u and we do not require the function $i \mapsto u_i$ to lie in the class $\mathbb{L}^u = \bigcup_{\alpha \in \text{Ord}} \mathbb{L}_\alpha^u$. We would like to define our search tree S_α^u as a labelled subtree of $(\mathbb{L}_\alpha^u)^{<\omega}$, the tree of finite sequences with entries in \mathbb{L}_α^u . However, there is one technical obstruction: We will later need a primitive recursive notion of \mathbb{L}^u -rank. This is problematic, for not even membership in the class \mathbb{L}^u seems to be primitive recursively decidable. To fix this we replace \mathbb{L}_α^u by its ranked version

$$\mathbf{L}_\alpha^u = \begin{cases} \{\langle 0, a \rangle \mid a \in \mathbb{L}_0^u\} & \text{if } \alpha = 0, \\ \{\langle \beta, a \rangle \mid \beta < \alpha \text{ minimal with } a \in \mathbb{L}_{\beta+1}^u\} & \text{if } \alpha > 0. \end{cases}$$

Note that $\mathbf{L}^u = \bigcup_{\alpha \in \text{Ord}} \mathbf{L}_\alpha^u$ is now a primitive recursive class. It is trivial to define a rank function

$$|\cdot|_{\mathbf{L}}^u : \mathbf{L}^u \rightarrow \text{Ord}, \quad |\langle \beta, a \rangle|_{\mathbf{L}}^u = \beta$$

and a projection

$$\text{pr}_{\mathbf{L}}^u : \mathbf{L}^u \rightarrow \mathbb{L}^u, \quad \text{pr}_{\mathbf{L}}^u(\langle \beta, a \rangle) = a$$

such that we have

$$|c|_{\mathbf{L}}^u = \min\{\alpha \in \text{Ord} \mid \text{pr}_{\mathbf{L}}^u(c) \in \mathbb{L}_{\alpha+1}^u\}$$

and

$$|c|_{\mathbf{L}}^u = \min\{\alpha \in \text{Ord} \mid c \in \mathbf{L}_{\alpha+1}^u\}$$

for all $c \in \mathbf{L}^u$. Note that we have $\mathbf{L}_\alpha^u \subseteq \mathbf{L}_\beta^u$ for $\alpha < \beta$ (if $\alpha = 0$ use $\mathbb{L}_0^u \subseteq \mathbb{L}_1^u$), as well as $\mathbf{L}_\lambda^u = \bigcup_{\alpha < \lambda} \mathbf{L}_\alpha^u$ for any limit ordinal λ . The projection $\text{pr}_{\mathbf{L}}^u : \mathbf{L}^u \rightarrow \mathbb{L}^u$ is bijective but has no primitive recursive inverse. However, given a bound $\alpha > 0$ with $a \in \mathbb{L}_\alpha^u$ we can primitive recursively compute the minimal ordinal $\beta < \alpha$ with $a \in \mathbb{L}_{\beta+1}^u$, leading to $\langle \beta, a \rangle \in \mathbf{L}_\alpha^u$. Misusing notation we will often write a at the place of $\langle \beta, a \rangle$; the first component can be “recovered” by the notation $\beta = |a|_{\mathbf{L}}^u$. In particular, since u is equal to \mathbb{L}_0^u we can view each $u_i \in u$ as the element $\langle 0, u_i \rangle$ of \mathbf{L}_α^u (for any α). It will be convenient to assume $0, 1 \in u$, for then we have $\langle 0, 0 \rangle, \langle 0, 1 \rangle \in \mathbb{L}_0^u$ (we will use $0, 1$ as markers for the two conjuncts / disjuncts of a formula).

As stated above, the search tree S_α^u will be a subtree of $(\mathbf{L}_\alpha^u)^{<\omega}$. Each node of S_α^u will be labelled by an \mathbf{L}_α^u -sequent, a notion that we shall define next: By a formula (of the object language) we shall mean a first-order formula with relation symbols \in and $=$. As common in proof theory we only consider formulas in negation normal form: These are built from negated and unnegated prime formulas by the connectives \wedge and \vee and the quantifiers \forall and \exists . To negate a formula one pushes negation down to the level of prime formulas (applying de Morgan’s laws) and deletes any double negations. Other connectives will be used as abbreviations, such that e.g. $\varphi \rightarrow \psi$ stands for $\neg\varphi \vee \psi$. An occurrence of a quantifier is called

bounded if it is of the form $\forall_x(x \in y \rightarrow \dots)$ resp. $\exists_x(x \in y \wedge \dots)$, where y is not the variable x . We will abbreviate this as $\forall_{x \in y} \dots$ resp. $\exists_{x \in y} \dots$ but we shall not consider bounded quantifiers as separate quantifiers in their own right; rather, they are bounded occurrences of normal quantifiers. A Δ_0 -formula is a formula in which all quantifiers are bounded. Formulas may contain arbitrary sets as parameters. By an \mathbf{L}_α^u -formula we mean a closed formula with parameters from \mathbf{L}_α^u . Note that for parameters in \mathbf{L}^u it is really the projection into \mathbb{L}^u that counts: e.g. the \mathbf{L}_α^u -formula $\langle \beta_0, a_0 \rangle \in \langle \beta_1, a_1 \rangle$ should really be interpreted as the formula $a_0 \in a_1$, together with information on the ranks of the sets $a_0, a_1 \in \mathbb{L}^u$. An \mathbf{L}_α^u -sequent is a finite sequence of \mathbf{L}_α^u -formulas. As usual we write Γ, φ for the sequent that arises from Γ by appending the formula φ as last entry. Concerning semantics, we have a primitive recursive notion of satisfaction of a formula in a set model $(m, \in|_{m \times m})$ (cf. [Bar75, Section III.1]). In particular we get a primitive recursive truth predicate for Δ_0 -formulas with parameters. We stress once more that the parameters of an \mathbf{L}_α^u -formula must be projected into \mathbb{L}_α^u before its satisfaction or truth are evaluated.

As a final ingredient for the definition of S_α^u we need an enumeration $\langle \theta_k \rangle_{k \in \omega}$ of the axioms of Kripke-Platek set theory with infinity, excluding the instances of foundation (which will hold in any transitive model). To give a concrete description of the axioms, recall the usual Δ_0 -formula expressing that a given set is a limit ordinal (cf. [Bar75, Chapter I]: there the formula has complexity Δ_1 , but only because of the implementation of urelements). Then $\langle \theta_k \rangle_{k \in \omega}$ lists the axioms

$$\begin{aligned}
(\text{Equality}) & \quad \forall_x \forall_{x'} \forall_y \forall_{y'} (x = x' \wedge y = y' \wedge x \in y \rightarrow x' \in y') \\
(\text{Extensionality}) & \quad \forall_x \forall_y (\forall_{z \in x} z \in y \wedge \forall_{z \in y} z \in x \rightarrow x = y) \\
(\text{Pairing}) & \quad \forall_x \forall_y \exists_z (x \in z \wedge y \in z) \\
(\text{Union}) & \quad \forall_x \exists_y \forall_{z \in x} \forall_{z' \in z} z' \in y \\
(\text{Infinity}) & \quad \exists_x \text{“}x \text{ is a limit ordinal”}
\end{aligned}$$

and the instances of the axiom schemata

$$\begin{aligned}
(\Delta_0\text{-separation}) & \quad \forall_{v_1} \dots \forall_{v_k} \forall_x \exists_y (\forall_{z \in x} (\theta(x, z, v_1, \dots, v_k) \rightarrow z \in y) \wedge \\
& \quad \wedge \forall_{z \in y} (z \in x \wedge \theta(x, z, v_1, \dots, v_k))) \\
(\Delta_0\text{-collection}) & \quad \forall_{v_1} \dots \forall_{v_k} \forall_x (\forall_{y \in x} \exists_z \theta(x, y, z, v_1, \dots, v_k) \rightarrow \\
& \quad \rightarrow \exists_w \forall_{y \in x} \exists_{z \in w} \theta(x, y, z, v_1, \dots, v_k))
\end{aligned}$$

for Δ_0 -formulas θ . It will be convenient to make two restrictions on the formula θ in the axiom schemata: Firstly, we fix some global bound on the number k of parameters in Δ_0 -separation and Δ_0 -collection axioms. This is a harmless restriction because all other instances can be derived via an encoding of tuples. Secondly, we require the formula θ in a Δ_0 -collection axiom to be a disjunction. This ensures that the existential quantifier in the subformula $\exists_z \theta$ is unbounded. To deduce Δ_0 -collection for an arbitrary Δ_0 -formula θ one replaces θ by the equivalent disjunction $z \neq z \vee \theta$. Now we are ready to define the desired search trees:

Definition 1.1. Given a transitive set $u = \{u_i \mid i \in \omega\} \supseteq \{0, 1\}$ and an ordinal α we define a search tree $S_\alpha^u \subseteq (\mathbf{L}_\alpha^u)^{<\omega}$ and a labelling $l : S_\alpha^u \rightarrow \text{“}\mathbf{L}_\alpha^u\text{-sequents”}$ by recursion on sequences $\sigma \in (\mathbf{L}_\alpha^u)^{<\omega}$. In the base case $\sigma = \langle \rangle$ we set

$$\langle \rangle \in S_\alpha^u \quad \text{and} \quad l(\langle \rangle) = \langle \rangle.$$

In the recursion step we assume that $\sigma \in S_\alpha^u$ holds. If σ has even length $2k$ then we add the negation of an axiom of Kripke-Platek set theory, setting

$$\sigma \frown a \in S_\alpha^u \Leftrightarrow a = 0 \quad \text{and} \quad l(\sigma \frown 0) = l(\sigma), \neg\theta_k.$$

If the length of σ is odd we analyze the previous sequent: If $l(\sigma)$ contains a true Δ_0 -formula then we stipulate that σ is a leaf of S_α^u . If $l(\sigma)$ consist of false (negated) prime formulas we set

$$\sigma \frown a \in S_\alpha^u \Leftrightarrow a = 0 \quad \text{and} \quad l(\sigma \frown 0) = l(\sigma).$$

Otherwise we write $l(\sigma) = \Gamma, \varphi, \Gamma'$ such that Γ consists of (negated) prime formulas and φ is not a (negated) prime formula. For later reference we call φ the redex of $l(\sigma)$. The recursion step depends on the form of φ as follows:

If...	... then ...
$\varphi \equiv \psi_0 \wedge \psi_1$	$\sigma \frown a \in S_\alpha^u$ iff $a \in \{0, 1\}$, and $l(\sigma \frown i) = \Gamma, \Gamma', \varphi, \psi_i$ for $i = 0, 1$,
$\varphi \equiv \psi_0 \vee \psi_1$	$\sigma \frown a \in S_\alpha^u$ iff $a = 0$, and $l(\sigma \frown 0) = \Gamma, \Gamma', \varphi, \psi_i$ where $i = 0$ if ψ_0 does not already occur in $l(\sigma)$ and $i = 1$ otherwise,
$\varphi \equiv \forall_x \psi(x)$	$\sigma \frown a \in S_\alpha^u$ for all $a \in \mathbf{L}_\alpha^u$, and $l(\sigma \frown a) = \Gamma, \Gamma', \varphi, \psi(a)$,
$\varphi \equiv \exists_x \psi(x)$	$\sigma \frown a \in S_\alpha^u$ iff $a = 0$, and $l(\sigma \frown 0) = \Gamma, \Gamma', \varphi, \psi(b)$ where b is the first entry of the list $u_0, \sigma_0, \dots, u_{\text{dom}(\sigma)-1}, \sigma_{\text{dom}(\sigma)-1}, u_{\text{dom}(\sigma)}, u_{\text{dom}(\sigma)+1}, \dots$ such that $\psi(b)$ does not already occur in $l(\sigma)$.

Consider a function $f : \omega \rightarrow \mathbf{L}_\alpha^u$ and write $f[n] = \langle f(0), \dots, f(n-1) \rangle$ for $n \in \omega$. If $f[n] \in S_\alpha^u$ holds for all $n \in \omega$ then f is called a branch of S_α^u . We say that a formula occurs on f if it occurs in some sequent $l(f[n])$. The construction of search trees ensures the following crucial properties:

Lemma 1.2. *For any branch f of the search tree S_α^u the following holds:*

- (a) Any parameter in a formula on f lies in $\text{rng}(f) \cup u$.
- (b) Any (negated) prime formula on f is false.
- (c) If $\psi_0 \wedge \psi_1$ occurs on f then either ψ_0 or ψ_1 occurs on f .
- (d) If $\psi_0 \vee \psi_1$ occurs on f then both ψ_0 and ψ_1 occur on f .
- (e) If $\forall_x \psi(x)$ occurs on f then $\psi(b)$ occurs on f for some $b \in \text{rng}(f)$.
- (f) If $\exists_x \psi(x)$ occurs on f then $\psi(b)$ occurs on f for all $b \in \text{rng}(f) \cup u$.

Proof. (a) By induction on n we show that all parameters that occur in the sequent $l(f[n])$ lie in $\text{rng}(f[n]) \cup u$: For $n = 0$ we have $l(f[0]) = \langle \rangle$ and thus no parameters. In the induction step we consider $f[n+1] = f[n] \frown f(n)$: If $n = 2k$ is even then the only new formula in $l(f[n+1])$ is the negated axiom $\neg\theta_k$, which contains no parameters. Now assume that n is odd. The first interesting case is that of a redex $\varphi \equiv \forall_x \psi(x)$. Then $l(f[n+1])$ contains the new formula $\psi(f(n))$. As the formula $\forall_x \psi(x)$ occurs in $l(f[n])$ its parameters lie in $\text{rng}(f[n]) \cup u$ by induction hypothesis. The only new parameter in the formula $\psi(f(n))$ is $f(n)$, which is indeed an element of $\text{rng}(f[n+1])$. The other interesting case is a redex $\varphi \equiv \exists_x \psi(x)$. Here $l(f[n])$ contains a new parameter from the list $u_0, f(0), u_1, \dots, f(n-1), u_n, u_{n+1}, u_{n+2}, \dots$,

which is thus an element of $\text{rng}(f[n]) \cup u$.

(b) Aiming at a contradiction, assume that $l(f[n])$ contains a true (negated) prime formula. By construction this means that $f[n]$ is a leaf of the search tree, contradicting the assumption that f is a branch.

(c) Similar to (e), and easier.

(d) Similar to (f), and easier.

(e) Assume that $\forall_x \psi(x)$ is a formula in $l(f[n])$. By construction of the search tree $\forall_x \psi(x)$ will be the redex of $l(f[m])$ for some $m \geq n$ (a formula to the right of the redex moves a position to the left in each odd step). Then the formula $\psi(f(m))$ occurs in $l(f[m+1])$, again by construction.

(f) As in (e) we may assume that $\exists_x \psi(x)$ is the redex of $l(f[m])$. By construction the formula $\psi(u_0)$ occurs in $l(f[m+1])$, and so does $\exists_x \psi(x)$. Now observe that $\psi(u_0)$ and $\exists_x \psi(x)$ remain in $l(f[m'])$ for all $m' \geq m+1$. Pick an m' such that $\exists_x \psi(x)$ is again the redex of $l(f[m'])$. Since $\psi(u_0)$ is already contained in $l(f[m'])$ we may conclude that $\psi(f(0))$ occurs in $l(f[m'+1])$. Inductively we can verify that $\psi(b)$ occurs on f for any b in the list $u_0, f(0), u_1, f(1), \dots$, which enumerates the set $\text{rng}(f) \cup u$. \square

To understand the following proposition, recall that $\text{pr}_{\mathbf{L}}^u$ projects the ranked constructible hierarchy \mathbf{L}^u onto the usual hierarchy \mathbb{L}^u , and that \mathbf{L}_α^u -formulas are to be evaluated under this projection.

Proposition 1.3. *Assume that f is a branch of the search tree S_α^u . Any formula that occurs on f is false in the structure $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$.*

Proof. We establish the claim by induction on the height of formulas. Note that the formulas on f form a set, so that the induction statement is primitive recursive. For a (negated) prime formula the claim holds by the previous lemma. Now let us consider a formula $\forall_x \psi(x)$ that occurs on f . By the lemma $\psi(b)$ occurs on f for some $b \in \text{rng}(f)$. The induction hypothesis tells us that $\psi(b)$ is false in the structure $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$; thus $\forall_x \psi(x)$ must also be false in this structure. Next, consider a formula $\exists_x \psi(x)$ on f . The lemma tells us that $\psi(b)$ occurs on f for all $b \in \text{rng}(f) \cup u$. By the induction hypothesis $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$ does not satisfy any of these formulas. Thus it cannot satisfy $\exists_x \psi(x)$ either. The remaining cases are similar and easier. \square

Corollary 1.4. *If f is a branch of the search tree S_α^u then $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$ is a model of Kripke-Platek set theory.*

Proof. Recall that $\langle \theta_k \rangle_{k \in \omega}$ enumerates the Kripke-Platek axioms, except for foundation. By construction of the search tree all formulas $\neg \theta_k$ occur on f . The proposition tells us that $\neg \theta_k$ is false in $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$. Thus this structure satisfies all axioms θ_k . As the satisfaction relation is primitive recursive, foundation for arbitrary formulas in $(\text{rng}(\text{pr}_{\mathbf{L}}^u \circ f) \cup u, \in)$ reduces to foundation for primitive recursive predicates in the meta-theory. \square

Recall that transitive models of Kripke-Platek set theory are also called admissible sets.

Corollary 1.5. *If the search tree S_α^u has a branch then there is an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$.*

Proof. The previous corollary provides a model in which \in is interpreted by the actual membership relation. In particular this is a model of extensionality. Thus it is isomorphic to its Mostowski collapse. As u is transitive it is still contained in the collapsed model. \square

If u is hereditarily countable we can apply the same construction to the transitive closure of $\{u\}$, to obtain an admissible set \mathbb{A} with $u \in \mathbb{A}$. To conclude that u is indeed contained in an admissible set one must exclude the case that the search trees S_α^u are well-founded for all ordinals α . This will require an additional assumption, namely a “higher well-ordering principle” to be described in the next section. In the rest of this section we exhibit some further properties of the trees S_α^u , which will link them to such well-ordering principles.

First, let us clarify our notion of well-ordering: We say that $<$ well-orders a if every non-empty subset of a has a $<$ -minimal element. Below we will relate this to the existence of infinitely decreasing sequences (in general this requires some amount of choice). Now we want to define linear orderings on the search trees S_α^u . Note that the set u admits a well-ordering which is primitive recursive in the given enumeration $u = \{u_i \mid i \in \omega\}$. This can be extended to a primitive recursive family of compatible well-orderings $<_{\mathbf{L}_\alpha^u}$ on the stages \mathbf{L}_α^u of the (ranked) constructible hierarchy (cf. [Sim09, Lemma VII.4.19]; to show that these are well-orderings one constructs order embeddings $(\mathbf{L}_\alpha^u, <_{\mathbf{L}_\alpha^u}) \rightarrow \text{Ord}$, also by primitive recursion). Using these well-orderings we can define the Kleene-Brouwer ordering on the tree S_α^u , namely as

$$\sigma <_{S_\alpha^u} \tau \quad :\Leftrightarrow \quad \begin{cases} \text{either the sequence } \sigma \text{ properly extends the sequence } \tau, \\ \text{or } \sigma = \sigma_0 \frown a \frown \sigma' \text{ and } \tau = \sigma_0 \frown b \frown \tau' \text{ with } a <_{\mathbf{L}_\alpha^u} b. \end{cases}$$

Clearly $<_{S_\alpha^u}$ is a linear ordering on S_α^u . The promised connection with well-orderedness goes as follows:

Lemma 1.6. *If the search tree S_α^u has no infinite branch then its Kleene-Brouwer ordering $<_{S_\alpha^u}$ is a well-ordering.*

Proof. The first step is to show that the “choice function”

$$\min_{\mathbf{L}_\alpha^u}(a) := \text{“the } <_{\mathbf{L}_\alpha^u}\text{-minimal element of } a \cap \mathbf{L}_\alpha^u\text{”}$$

is primitive recursive. Indeed the set

$$\{b \in a \cap \mathbf{L}_\alpha^u \mid \neg \exists y \in a \cap \mathbf{L}_\alpha^u \ y <_{\mathbf{L}_\alpha^u} b\}$$

can be computed by a primitive recursive function. As $<_{\mathbf{L}_\alpha^u}$ is a well-ordering this set is a singleton (or empty, in which case we assign some default value), and we can extract its only element $\min_{\mathbf{L}_\alpha^u}(a)$. Now the claim of the lemma is shown by contraposition: Assume that $a \subseteq S_\alpha^u$ has no $<_{S_\alpha^u}$ -minimal element. Observe that we have $a \subseteq (\mathbf{L}_\alpha^u)^{<\omega} \subseteq \mathbf{L}_{\alpha+\omega}^u$. Thus we can use the choice function for subsets of $\mathbf{L}_{\alpha+\omega}^u$ to define a $<_{S_\alpha^u}$ -descending sequence $g : \omega \rightarrow x$, setting

$$g(n+1) := \min_{\mathbf{L}_{\alpha+\omega}^u}(\{b \in a \mid b <_{S_\alpha^u} g(n)\}).$$

To transform g into an infinite branch f of S_α^u we recursively define

$$f(n) := \min_{\mathbf{L}_\alpha^u}(\{b \in \mathbf{L}_\alpha^u \mid f[n] \frown b \in S_\alpha^u \text{ and } f[n] \frown b \text{ lies below } g(m) \\ \text{for infinitely many } m \in \omega\}).$$

Here “ $f[n] \frown b$ lies below $g(m)$ ” means that the sequence $g(m)$ is an end-extension of the sequence $f[n] \frown b$. Note that the property “is an infinite subset of ω ” is primitive recursive. To conclude it suffices to show that the required sets b exist: Inductively we assume that $f[n]$ lies (strictly) below infinitely many nodes of the form $g(m)$. Define a strictly increasing sequence of numbers m_k such that $f[n]$ lies below all nodes $g(m_k)$. Let b_k be the unique set in \mathbf{L}_α^u such that $f[n] \frown b_k$ lies below $g(m_k)$. From $g(m_{k+1}) <_{S_\alpha^u} g(m_k)$ and the definition of the Kleene-Brouwer ordering we get $b_{k+1} \leq_{\mathbf{L}_\alpha^u} b_k$. As $<_{\mathbf{L}_\alpha^u}$ is well-founded there must be a bound K such that $b_k = b_K$ holds for all $k \geq K$. It follows that $f[n] \frown b_K$ lies below $g(m_k)$ for all $k \geq K$, so $b := b_K$ is as required for the definition of $f(n)$. \square

Next, let us observe that the linear orderings $(S_\alpha^u, <_{S_\alpha^u})$ are compatible:

Lemma 1.7. *For $\alpha < \beta$ we have $S_\alpha^u \subseteq S_\beta^u$ and $<_{S_\alpha^u} = <_{S_\beta^u} \cap (S_\alpha^u \times S_\alpha^u)$. If λ is a limit ordinal then we have $S_\lambda^u = \bigcup_{\alpha < \lambda} S_\alpha^u$.*

Proof. By induction on sequences $\sigma \in \mathbf{L}_\alpha^u$ one verifies $\sigma \in S_\alpha^u \Leftrightarrow \sigma \in S_\beta^u$; simultaneously one needs to check that the labels in the two trees coincide. Thus we have $S_\alpha^u = S_\beta^u \cap (\mathbf{L}_\alpha^u)^{<\omega}$, from which the claims are easily deduced. \square

It is important to observe that, given $\alpha < \beta$, the order $(S_\alpha^u, <_{S_\alpha^u})$ is *not* an initial segment of $(S_\beta^u, <_{S_\beta^u})$: Indeed the root $\langle \rangle$ is the biggest element of any S_α^u , and it already lies in S_0^u (cf. Example 2.5 below).

Remark 1.8. Recall Girard’s notion of (pre-)dilator [Gir81]. In particular a pre-dilator is a functor from the category of well-orders to the category of linear orders. To turn our construction of search trees

$$\alpha \mapsto (S_\alpha^u, <_{S_\alpha^u})$$

into such a functor we would have to assign an embedding $(S_\alpha^u, <_{S_\alpha^u}) \rightarrow (S_\beta^u, <_{S_\beta^u})$ to each order preserving map $\alpha \rightarrow \beta$. The previous lemma yields such an embedding for the inclusion map of α into $\beta > \alpha$. The obvious extension to arbitrary maps requires a functorial version of the constructible hierarchy (cf. the notion of β -proof in [Gir85, Section 6]). Dilators have the foundational advantage that they are finitistically meaningful (see [Gir81, Section 0.2.1]). Apart from that there is no need to work with dilators in the present study — interestingly enough, though, the embeddings $S_\alpha^u \subseteq S_\beta^u$ from the previous lemma will play a key role. Many ideas that we use come from Girard’s work on Π_2^1 -logic, even if we do not work with dilators and β -proofs in the strict sense.

To conclude this section, let us show that the rank function for the constructible hierarchy yields a rank function for the sequence $\alpha \mapsto S_\alpha^u$ of search trees:

Lemma 1.9. *The class $S^u := \bigcup_{\alpha \in \text{Ord}} S_\alpha^u$ is primitive recursive in the given enumeration of u . There is a primitive recursive rank function $|\cdot|_S^u : S^u \rightarrow \text{Ord}$ such that we have $|\sigma|_S^u = \min\{\alpha \in \text{Ord} \mid \sigma \in S_{\alpha+1}^u\}$ for all $\sigma \in S^u$.*

Proof. First define $|\cdot|_S^u$ on the full tree $(\mathbf{L}^u)^{<\omega}$, namely by

$$|\sigma|_S^u = \begin{cases} 0 & \text{if } \sigma = \langle \rangle, \\ \max\{|a_0|_{\mathbf{L}}^u, \dots, |a_n|_{\mathbf{L}}^u\} & \text{if } \sigma = \langle a_0, \dots, a_n \rangle. \end{cases}$$

Then

$$|\sigma|_S^u = \min\{\alpha \in \text{Ord} \mid \sigma \in (\mathbf{L}_{\alpha+1}^u)^{<\omega}\}$$

follows from the corresponding property of the ranked constructible hierarchy. In the proof of Lemma 1.7 we have seen $S^u \cap (\mathbf{L}_\alpha^u)^{<\omega} = S_\alpha^u$. Thus $\sigma \in S^u$ is equivalent to $\sigma \in S_{|\sigma|_S^u+1}^u$, which is a primitive recursive relation. Now we may restrict $|\cdot|_S^u$ to the class S^u , and $|\sigma|_S^u = \min\{\alpha \in \text{Ord} \mid \sigma \in S_{\alpha+1}^u\}$ follows from the above. \square

Depending on the situation it may be more intuitive to think of the compatible family $\alpha \mapsto S_\alpha^u$ of set-sized trees or of the single class-sized tree S^u .

2. A HIGHER BACHMANN-HOWARD CONSTRUCTION

In the previous section we have constructed, for each countable transitive set u , a family of search trees $\langle S_\alpha^u \rangle_{\alpha \in \text{Ord}}$ with the following property: If there is an ordinal α such that S_α^u is ill-founded then there is an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$. It remains to consider the case where all search trees S_α^u are well-founded. In this case the primitive recursive function $\alpha \mapsto S_\alpha^u$ is a *well-ordering principle*, in a sense to be defined below. The goal of this section is to define a notion of Bachmann-Howard ordinal relative to a well-ordering principle. The assertion that “the Bachmann-Howard ordinal relative to any given well-ordering principle exists” can itself be described as a *higher well-ordering principle*. In the following sections we will use this higher well-ordering principle to exclude the case that all S_α^u are well-founded. This will finally establish the existence of an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$.

We begin with a general notion of well-ordering principle:

Definition 2.1. Consider primitive recursive functions $T : (u, \alpha) \mapsto (T_\alpha^u, <_{T_\alpha^u})$ and $|\cdot|_T : (u, s) \mapsto |s|_T^u$. We say that T^u is a (ranked compatible) well-ordering principle, abbreviated as $\text{WOP}(T^u)$, if the following holds:

- (i) For every ordinal α the set T_α^u is well-ordered by $<_{T_\alpha^u}$.
- (ii) For $\alpha < \beta$ we have $T_\alpha^u \subseteq T_\beta^u$ and $<_{T_\alpha^u} = <_{T_\beta^u} \cap (T_\alpha^u \times T_\alpha^u)$; furthermore we have $T_\lambda^u = \bigcup_{\alpha < \lambda} T_\alpha^u$ for each limit ordinal λ .
- (iii) We have

$$|s|_T^u = \begin{cases} \min\{\alpha \in \text{Ord} \mid s \in T_{\alpha+1}^u\} & \text{if such an } \alpha \text{ exists,} \\ \{1\} & \text{otherwise.} \end{cases}$$

As the set $\{1\}$ is not an ordinal (which is its sole purpose) this does, in particular, make the class $T^u = \bigcup_{\alpha \in \text{Ord}} T_\alpha^u$ primitive recursive.

The above is in fact a definition scheme: For fixed function symbols T and $|\cdot|_T$ we obtain a statement $\text{WOP}(T^u)$ with parameter u . Observe that $\text{WOP}(T^u)$ is a Π_1 -formula in the language of primitive recursive set theory. Also, note that $s \in T_\alpha^u$ implies $|\cdot|_T^u < \alpha$ for all $\alpha > 0$, by the minimality of the rank and the “continuity” in clause (ii). Let us now define a notion of collapse for ranked well-orderings:

Definition 2.2. Adding to Definition 2.1, a function $\vartheta : T_\alpha^u \rightarrow \alpha$ is called a Bachmann-Howard collapse of T_α^u , abbreviated as $\vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$, if the following holds for all $s, t \in T_\alpha^u$:

- (i) $|s|_T^u < \vartheta(s)$,
- (ii) if $s <_{T_\alpha^u} t$ and $|s|_T^u < \vartheta(t)$ then $\vartheta(s) < \vartheta(t)$.

Observe that $\vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$ is a primitive recursive property of ϑ, α, u . We shall motivate the definition in a moment, but let us first use it to state our higher well-ordering principle:

Definition 2.3. The higher Bachmann-Howard principle for T is the statement

$$\text{BH}(T) := \forall_u (\text{WOP}(T^u) \rightarrow \exists_\alpha \exists_\vartheta \vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha).$$

An ordinal α with $\exists_\vartheta \vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$ is called a Bachmann-Howard ordinal for T^u .

Observe that $\text{WOP}(T^u) \rightarrow \exists_\alpha \exists_\vartheta \vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$ is a Σ_1 -statement in the language of primitive recursive set-theory. Thus $\text{BH}(T)$ is a Π_2 -statement. The notion of Bachmann-Howard collapse is motivated by Rathjen's ordinal notation system for the Bachmann-Howard ordinal (see e.g. [RV15]). However, we can also give some motivation without recourse to this background: First, note that condition (i) of Definition 2.2 excludes the trivial solution $\vartheta(s) = 0$ for all $s \in T_\alpha^u$, which would fulfill condition (ii). Also note that (i) entails the implication

$$\vartheta(t) \leq |s|_T^u \Rightarrow \vartheta(t) < \vartheta(s),$$

familiar from Rathjen's ordinal notation system. Let us record an easy consequence:

Lemma 2.4. *If T^u is a well-ordering principle then any Bachmann-Howard collapse $\vartheta : T_\alpha^u \xrightarrow{\text{BH}} \alpha$ is injective.*

Proof. Consider arbitrary elements $s, t \in T_\alpha^u$. As T_α^u is linearly ordered we may assume $s <_{T_\alpha^u} t$. Now distinguish the following cases: If we have $|s|_T^u < \vartheta(t)$ then condition (ii) of Definition 2.2 implies $\vartheta(s) < \vartheta(t)$. If, on the other hand, we have $\vartheta(t) \leq |s|_T^u$ then we get $\vartheta(t) < \vartheta(s)$, as we have just seen. \square

Note that any Bachmann-Howard collapse preserves the ordering between elements of the same rank: If $|s|_T^u \leq |t|_T^u$ then condition (i) of Definition 2.2 yields $|s|_T^u < \vartheta(t)$. Together with condition (ii) this implies $\vartheta(s) < \vartheta(t)$. On the other hand $|s|_T^u \leq |t|_T^u$ is not a necessary condition for $\vartheta(s) < \vartheta(t)$. We will later see that the weaker condition $|s|_T^u < \vartheta(t)$ plays a crucial role. The following explains why we do not require ϑ to be completely order preserving:

Example 2.5. Consider the well-ordering principle $T_\alpha := \alpha \cup \{\star\}$ with the usual ordering on α and \star as biggest element. Then $T = \bigcup_{\alpha \in \text{Ord}} T_\alpha$ is the class of all ordinals with a maximal element added. We have $|\beta|_T = \beta$ and $|\star|_T = 0$. The order type of T_α is $\alpha + 1$, so there can be no order preserving map $\vartheta : T_\alpha \rightarrow \alpha$. However, if we demand $\vartheta(s) < \vartheta(t)$ only under the side condition $|s|_T < \vartheta(t)$, then such a map exists for $\alpha = \omega \cdot 2$: Set $\vartheta(\beta) = \beta + 1$ and $\vartheta(\star) = \omega$. Indeed, $|\beta|_T < \vartheta(\star)$ now implies $\beta < \omega$ and thus $\vartheta(\beta) < \omega = \vartheta(\star)$. Also observe that $\vartheta(\star) \geq \omega$ must hold for any Bachmann-Howard collapse $\vartheta : T_{\omega \cdot 2} \xrightarrow{\text{BH}} \omega \cdot 2$: First, we must have $0 = |\star|_T < \vartheta(\star)$. Inductively we assume $|n|_T = n < \vartheta(\star)$ and infer $n = |n|_T < \vartheta(n) < \vartheta(\star)$, which implies $n + 1 < \vartheta(\star)$.

Having seen the example, the reader may rightly ask whether each well-ordering principle allows for a Bachmann-Howard collapse. In Section 6 we will show that a Bachmann-Howard collapse can be constructed on the basis of an admissible set. The following foreshadows this construction, but in a strong meta-theory:

Remark 2.6. Consider a well-ordering principle T^u with $u \in \mathbb{L}_{\aleph_1}$ (where \aleph_1 is the first uncountable cardinal). As T is a primitive recursive function we have $T_\alpha^u \in \mathbb{L}_{\aleph_1}$ for each $\alpha < \aleph_1$; in particular the sets T_α^u are countable. We define a Bachmann-Howard collapse $\vartheta : T_{\aleph_1}^u \rightarrow \aleph_1$ by recursion over the well-ordering $T_{\aleph_1}^u$. Assuming

that $\vartheta(s)$ is already defined for all $s <_{T_{\aleph_1}^u} t$ let us construct sets $C_n(t, \alpha) \subseteq \aleph_1$ by recursion over $n \in \omega$, for all $\alpha < \aleph_1$:

- $C_0(t, \alpha) = \alpha \cup \{|t|_T^u\}$,
- $C_{n+1}(t, \alpha) = C_n(t, \alpha) \cup \{\vartheta(s) \mid s <_{T_{\aleph_1}^u} t \text{ and } |s|_T^u \in C_n(t, \alpha)\}$.

Now set $C(t, \alpha) := \bigcup_{n \in \omega} C_n(t, \alpha)$ and

$$\vartheta(t) := \min\{\alpha < \aleph_1 \mid C(t, \alpha) \subseteq \alpha\}.$$

We must verify that such an α exists: For each countable β there are only countably many $s \in T_{\aleph_1}^u$ with $|s|_T^u = \beta$, because we have $s \in T_{|s|_T^u+1}^u$. Thus if $C_n(t, \alpha)$ is countable then so is the set

$$\bigcup_{\beta \in C_n(t, \alpha)} \{\vartheta(s) \mid |s|_T^u = \beta\}.$$

Inductively it follows that all $C_n(t, \alpha)$ are countable. So $C(t, \alpha)$ is countable as well. We can thus construct a sequence $0 = \alpha_0 < \alpha_1 < \dots < \aleph_1$ with $C(t, \alpha_n) \subseteq \alpha_{n+1}$. Set $\alpha := \sup_{n \in \omega} \alpha_n < \aleph_1$. It is easy to verify

$$C(t, \alpha) = \bigcup_{n \in \omega} C(t, \alpha_n) \subseteq \bigcup_{n \in \omega} \alpha_{n+1} = \alpha,$$

as required. Conditions (i) and (ii) from Definition 2.2 are readily deduced: We have $|s|_T^u \in C(s, \vartheta(s)) \subseteq \vartheta(s)$. Also, $|s|_T^u < \vartheta(t)$ implies $|s|_T^u \in C(t, \vartheta(t))$. Together with $s <_{T_{\aleph_1}^u} t$ this yields $\vartheta(s) \in C(t, \vartheta(t)) \subseteq \vartheta(t)$. The proof-theorist will have noticed that the given argument is very similar to the usual construction of the Bachmann-Howard ordinal.

We want to show that our higher Bachmann-Howard principle implies the existence of admissible sets. Let us compare this claim with some known results:

Remark 2.7. Recall axiom beta (see e.g. [Bar75, Definition I.9.5]), which states that any well-founded relation can be collapsed to the \in -relation. It is easy to deduce axiom beta for linear orderings from our higher Bachmann-Howard principle: Define $T_\alpha^{(u, <_u)} := (u, <_u)$ and

$$|s|_T^{(u, <_u)} := \begin{cases} 0 & \text{if } s \in u, \\ \{1\} & \text{otherwise.} \end{cases}$$

If $(u, <_u)$ is a well-ordering then $T^{(u, <_u)}$ is a well-ordering principle. The higher Bachmann-Howard principle provides a Bachmann-Howard collapse $\vartheta : u \rightarrow \alpha$ for some ordinal α . In the present case ϑ is fully order preserving, because all elements of u receive the same rank zero. Thus axiom beta is established. This observation sparks the following question: Can we construct admissible sets on the basis of axiom beta alone, making our higher Bachmann-Howard principle redundant? Indeed, axiom beta is powerful in the presence of Δ -separation: As axiom beta turns well-foundedness into a Δ -property we see that the definable well-orderings of the natural numbers form a set. Then Σ -collection ensures the existence of the Church-Kleene ordinal, which is well-known to be admissible. Also, the combination of axiom beta and Σ -collection implies Δ_2^1 -comprehension (see [Poh98, Theorem 3.3.4.7]). In particular we get Π_1^1 -comprehension, which is equivalent to the existence of countable admissible sets (see [Poh98, Theorem 3.3.3.5] and [Jäg86, Lemma 7.5]). To summarize, our higher Bachmann-Howard principle does not add more strength

than the bare axiom beta over a base theory that contains Σ -collection (such as Kripke-Platek set theory). For a base theory that does not contain Σ -collection and Δ -separation the situation can be quite different: Consider for example the set-theoretic version of **ATR**₀ introduced by Simpson (see [Sim09, Section VII.3]). This theory contains axiom beta but does not prove Π_1^1 -comprehension or the existence of admissible sets. Over such base theories the higher Bachmann-Howard principle is thus a genuine strengthening of axiom beta (anticipating our construction of admissible sets based on a Bachmann-Howard collapse). We remark that theories without Δ -separation and Σ -collection are particularly interesting in the context of reverse mathematics. Many interesting questions remain open: What precisely is responsible for the strength of the higher Bachmann-Howard principle? Can we weaken the conditions on a Bachmann-Howard collapse, e.g. by replacing conditions (i,ii) of Definition 2.2 with the weaker implication $s <_{T_\alpha^u} t \wedge |s|_T^u = |t|_T^u \Rightarrow \vartheta(s) < \vartheta(t)$? Can we find a set theoretic or recursion theoretic proof of our result? Assuming the higher Bachmann-Howard principle, is there a “direct” construction of the Church-Kleene ordinal?

Recall the construction of search trees S_α^u from the previous section. We will not apply the higher Bachmann-Howard principle to the search trees themselves but rather to a modified well-ordering principle $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$, which combines ideas from [AR09, Definition 2.1] and [Rat92a, Definition 4.1]. In the following term systems the reader may interpret Ω as the ordinal ω^α or, alternatively, as the class of all ordinals: The first interpretation helps to understand each term system individually, while the second clarifies the relation of the term systems for different values of α . The terms ε_σ should be imagined as ε -numbers above Ω .

Definition 2.8. For each countable transitive set $u = \{u_i \mid i \in \omega\}$ and each ordinal α we define a set of terms $\varepsilon(S_{\omega^\alpha}^u)$ and an order relation $<_{\varepsilon(S_{\omega^\alpha}^u)}$ by the following simultaneous recursion (which will be justified below):

- (i) The symbol 0 is a term in $\varepsilon(S_{\omega^\alpha}^u)$.
- (ii) For each $\sigma \in S_{\omega^\alpha}^u$ the symbol ε_σ is a term in $\varepsilon(S_{\omega^\alpha}^u)$.
- (iii) Given terms $s_0, \dots, s_n \in \varepsilon(S_{\omega^\alpha}^u)$ and ordinals $0 < \beta_0, \dots, \beta_n < \omega^\alpha$ the expression

$$\Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n$$

is also a term in $\varepsilon(S_{\omega^\alpha}^u)$, provided that the following holds: If we have $n = 0$ and $\beta_0 = 1$ then s_0 may not be of the form ε_σ . If we have $n > 0$ then we require $s_{i+1} <_{\varepsilon(S_{\omega^\alpha}^u)} s_i$ for all $i < n$.

We stipulate that $s <_{\varepsilon(S_{\omega^\alpha}^u)} t$ holds if and only if one of the following is satisfied:

- (i) $s = 0$ and $t \neq 0$ (equality of terms),
- (ii) $s = \varepsilon_\sigma$ and one of the following holds:
 - $t = \varepsilon_\tau$ for some $\tau \in S_{\omega^\alpha}^u$ with $\sigma <_{S_{\omega^\alpha}^u} \tau$,
 - $t = \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m$ and $s <_{\varepsilon(S_{\omega^\alpha}^u)} t_0$ or $s = t_0$,
- (iii) $s = \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n$ and one of the following holds:
 - $t = \varepsilon_\tau$ and $s_0 <_{\varepsilon(S_{\omega^\alpha}^u)} t$,
 - $t = \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m$ and either
 - $n < m$ and $\langle s_i, \beta_i \rangle = \langle t_i, \gamma_i \rangle$ for all $i \leq n$, or
 - there is a $j \leq \min\{n, m\}$ such that we have either $s_j <_{\varepsilon(S_{\omega^\alpha}^u)} t_j$ or $s_j = t_j$ and $\beta_j < \gamma_j$, and $\langle s_i, \beta_i \rangle = \langle t_i, \gamma_i \rangle$ holds for all $i < j$.

The given definition can be justified as follows: First construct a preliminary term system $\varepsilon^0(S_{\omega_\alpha}^u)$ which is defined as above but contains all terms of the form

$$\Omega^{s_0} \cdot \beta_0 + \cdots + \Omega^{s_n} \cdot \beta_n,$$

regardless of the condition $s_{i+1} <_{\varepsilon(S_{\omega_\alpha}^u)} s_i$. Clearly $\varepsilon^0(S_{\omega_\alpha}^u)$ can be constructed by primitive recursion (just as the set of formulas with parameters from a given set). Also by primitive recursion we can define the length of terms in $\varepsilon^0(S_{\omega_\alpha}^u)$, setting

$$\text{len}(0) := \text{len}(\varepsilon_\sigma) := 0,$$

$$\text{len}(\Omega^{s_0} \cdot \beta_0 + \cdots + \Omega^{s_n} \cdot \beta_n) := \text{len}(s_0) + \cdots + \text{len}(s_n) + n + 1.$$

Then the conditions in Definition 2.8 single out a subset $\varepsilon(S_{\omega_\alpha}^u) \subseteq \varepsilon^0(S_{\omega_\alpha}^u)$ and a relation $<_{\varepsilon(S_{\omega_\alpha}^u)} \subseteq \varepsilon(S_{\omega_\alpha}^u) \times \varepsilon(S_{\omega_\alpha}^u)$ in the following way:

- To determine whether we have $s \in \varepsilon(S_{\omega_\alpha}^u)$ we only need to check $t \in \varepsilon(S_{\omega_\alpha}^u)$ for $\text{len}(t) < \text{len}(s)$, and $t <_{\varepsilon(S_{\omega_\alpha}^u)} t'$ for $\text{len}(t) + \text{len}(t') < \text{len}(s)$.
- To determine whether we have $s <_{\varepsilon(S_{\omega_\alpha}^u)} t$ we need to check $s' \in \varepsilon(S_{\omega_\alpha}^u)$ for $\text{len}(s') \leq \text{len}(s) + \text{len}(t)$, and $s' <_{\varepsilon(S_{\omega_\alpha}^u)} t'$ for $\text{len}(s') + \text{len}(t') < \text{len}(s) + \text{len}(t)$.

It follows that there is a primitive recursive function which constructs $\varepsilon(S_{\omega_\alpha}^u)$ and $<_{\varepsilon(S_{\omega_\alpha}^u)}$ from α and the given enumeration of u . Our next goal is to show that $\alpha \mapsto \varepsilon(S_{\omega_\alpha}^u)$ is a well-ordering principle if all the search trees $S_{\omega_\alpha}^u$ are well-founded. The following is a first step:

Lemma 2.9. *For all u, α the relation $<_{\varepsilon(S_{\omega_\alpha}^u)}$ is a linear ordering of $\varepsilon(S_{\omega_\alpha}^u)$.*

Proof. Recall that $<_{S_{\omega_\alpha}^u}$ is a linear ordering of $S_{\omega_\alpha}^u$. The claim for $<_{\varepsilon(S_{\omega_\alpha}^u)}$ follows by tedious but straightforward inductions on the length of terms (see above): By induction on $\text{len}(s)$ one shows that $s <_{\varepsilon(S_{\omega_\alpha}^u)} s$ is false. To see that $s <_{\varepsilon(S_{\omega_\alpha}^u)} t$ and $t <_{\varepsilon(S_{\omega_\alpha}^u)} r$ imply $s <_{\varepsilon(S_{\omega_\alpha}^u)} r$ one argues by induction on $\text{len}(s) + \text{len}(t) + \text{len}(r)$. Finally, by induction on $\text{len}(s) + \text{len}(t)$ one shows that one of the alternatives $s <_{\varepsilon(S_{\omega_\alpha}^u)} t$, $s = t$ (equality of terms) and $t <_{\varepsilon(S_{\omega_\alpha}^u)} s$ must hold. \square

Primitive recursive set theory does not show that any well-ordering is isomorphic to an ordinal. Nevertheless it is instructive to assume that we have an order embedding $c : S_{\omega_\alpha}^u \rightarrow \text{Ord}$, and to consider the following construction: Pick an ε -number $\varepsilon_\eta \geq \omega^\alpha$. Now define a function $o : \varepsilon(S_{\omega_\alpha}^u) \rightarrow \text{Ord}$ by

$$o(0) := 0,$$

$$o(\varepsilon_s) := \varepsilon_{\eta+1+c(s)},$$

$$o(\Omega^{s_0} \cdot \beta_0 + \cdots + \Omega^{s_n} \cdot \beta_n) := (\omega^{1+\alpha})^{o(s_0)} \cdot \beta_0 + \cdots + (\omega^{1+\alpha})^{o(s_n)} \cdot \beta_n.$$

By induction on $\text{len}(s) + \text{len}(t)$ one shows that $s <_{\varepsilon(S_{\omega_\alpha}^u)} t$ implies $o(s) < o(t)$. We have thus constructed an order embedding of $\varepsilon(S_{\omega_\alpha}^u)$ into the ordinals. As stated above, we do not in general have the function $c : S_{\omega_\alpha}^u \rightarrow \text{Ord}$ required for this interpretation at our disposal. Nevertheless we will be able to show that $\varepsilon(S_{\omega_\alpha}^u)$ is a well-ordering, provided that the same holds for $S_{\omega_\alpha}^u$. First, we need to extend addition and exponentiation to the full term system $\varepsilon(S_{\omega_\alpha}^u)$. Let us assume $\alpha > 0$ to have the coefficients $1, 2 < \omega^\alpha$ available. Exponentiation to the base Ω is easily defined, namely by

$$\Omega^s := \begin{cases} s & \text{if } s \text{ is of the form } \varepsilon_\sigma, \\ \Omega^s \cdot 1 & \text{otherwise.} \end{cases}$$

Iterated exponentiation is written as

$$\begin{aligned}\Omega_0^s &:= s, \\ \Omega_{n+1}^s &:= \Omega^{\Omega_n^s}.\end{aligned}$$

Extending addition to all terms in $\varepsilon(S_{\omega^\alpha}^u)$ is more tedious because we need to distinguish many cases. Luckily, the correct definition is evident if one thinks in terms of Cantor normal forms:

$$0 + s = s + 0 = s,$$

$$\varepsilon_\sigma + \varepsilon_\tau = \begin{cases} \varepsilon_\tau & \text{if } \varepsilon_\sigma < \varepsilon_\tau, \\ \Omega^{\varepsilon_\sigma} \cdot 2 & \text{if } \varepsilon_\sigma = \varepsilon_\tau, \\ \Omega^{\varepsilon_\sigma} \cdot 1 + \Omega^{\varepsilon_\tau} \cdot 1 & \text{if } \varepsilon_\sigma > \varepsilon_\tau, \end{cases}$$

$$\varepsilon_\sigma + (\Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m) = \begin{cases} \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } \varepsilon_\sigma < t_0, \\ \Omega^{t_0} \cdot (1 + \gamma_0) + \Omega^{t_1} \cdot \gamma_1 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } \varepsilon_\sigma = t_0, \\ \Omega^{\varepsilon_\sigma} \cdot 1 + \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } t_0 < \varepsilon_\sigma, \end{cases}$$

$$(\Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n) + \varepsilon_\tau = \begin{cases} \varepsilon_\tau & \text{if } s_0 < \varepsilon_\tau, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_{i-1}} \cdot \beta_{i-1} + \Omega^{s_i} \cdot (\beta_i + 1) & \text{if } s_i = \varepsilon_\tau, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_i} \cdot \beta_i + \Omega^{\varepsilon_\tau} \cdot 1 & \text{if } s_{i+1} < \varepsilon_\tau < s_i, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n + \Omega^{\varepsilon_\tau} \cdot 1 & \text{if } \varepsilon_\tau < s_n, \end{cases}$$

$$\begin{aligned} & (\Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n) + (\Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m) = \\ & = \begin{cases} \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } s_0 < t_0, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_{i-1}} \cdot \beta_{i-1} + \Omega^{s_i} \cdot (\beta_i + \gamma_0) + \Omega^{t_1} \cdot \gamma_1 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } s_i = t_0, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_i} \cdot \beta_i + \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } s_{i+1} < t_0 < s_i, \\ \Omega^{s_0} \cdot \beta_0 + \dots + \Omega^{s_n} \cdot \beta_n + \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m & \text{if } t_0 < s_n. \end{cases} \end{aligned}$$

In the last case distinction, observe that we have $\beta_i + \gamma_0 < \omega^\alpha$ because ω^α is additively closed. For $\alpha > 0$ the term system $\varepsilon(S_{\omega^\alpha}^u)$ is thus closed under addition. As in the usual ordinal notation systems (see e.g. [Sch77, Section V.14]) one can verify the expected relations:

Lemma 2.10. *Assume $\alpha > 0$. The following holds for all $r, s, t \in \varepsilon(S_{\omega^\alpha}^u)$:*

- (i) $s \leq \Omega^s$, and $s < t$ implies $\Omega^s < \Omega^t$,
- (ii) if $t < t'$ then $s + t < s + t'$ and $t + s \leq t' + s$,
- (iii) $s + (t + r) = (s + r) + t$,
- (iv) if $s < \Omega^t$ then $s + \Omega^t = \Omega^t$,

(v) if $s \leq t$ then we have $t = s + r$ for some r .

As promised, we can now show that well-orderedness is preserved:

Lemma 2.11. *If $(S_{\omega^\alpha}^u, <_{S_{\omega^\alpha}^u})$ is a well-ordering then so is $(\varepsilon(S_{\omega^\alpha}^u), <_{\varepsilon(S_{\omega^\alpha}^u)})$, for each $\alpha > 0$.*

Concerning the restriction $\alpha > 0$, we will later show that $\varepsilon(S_{\omega_0}^u)$ is a sub-ordering of $\varepsilon(S_{\omega_1}^u)$. Thus the well-foundedness of $\varepsilon(S_{\omega_0}^u)$ is also covered.

Proof. We adapt the argument from [Sch77, Lemma VIII.5] to our context (cf. also the result of [AR09]): As usual, well-foundedness is equivalent to induction, i.e. it suffices to establish

$$\forall_{s \in \varepsilon(S_{\omega^\alpha}^u)} (\forall_{t \in \varepsilon(S_{\omega^\alpha}^u)} (t <_{\varepsilon(S_{\omega^\alpha}^u)} s \rightarrow t \in a) \rightarrow s \in a) \rightarrow S_{\omega^\alpha}^u \subseteq a$$

for an arbitrary set a . Let us abbreviate

$$\text{Prog}(a) \equiv \forall_{s \in \varepsilon(S_{\omega^\alpha}^u)} (\forall_{t \in \varepsilon(S_{\omega^\alpha}^u)} (t <_{\varepsilon(S_{\omega^\alpha}^u)} s \rightarrow t \in a) \rightarrow s \in a).$$

In the rest of the proof we will drop the subscript of $<_{\varepsilon(S_{\omega^\alpha}^u)}$. Quantifiers with bound variable s, t or r are always restricted to $\varepsilon(S_{\omega^\alpha}^u)$. It is easy to see that any term in $\varepsilon(S_{\omega^\alpha}^u)$ is smaller than a term of the form $\Omega_n^{\varepsilon_\sigma+1}$. Thus it will be enough to show

$$\text{Prog}(a) \rightarrow \forall_{r < \Omega_n^{\varepsilon_\sigma+1}} r \in a$$

for all $\sigma \in S_{\omega^\alpha}^u$ and $n \in \omega$. We want to argue by induction on $(\sigma, n) \in S_{\omega^\alpha}^u \times \omega$, ordered alphabetically. The latter is a well-ordering since $S_{\omega^\alpha}^u$ is well-ordered by assumption. Now, induction over any well-ordering is available if the induction statement is primitive recursive (and thus, by separation, corresponds to a subset of the well-ordering). For fixed a the statement above is indeed primitive recursive; however it ceases to be primitive recursive if we quantify over all subsets $a \subseteq S_{\omega^\alpha}^u$. To gain flexibility while keeping the induction statement primitive recursive we introduce the primitive recursive “jump” function

$$J(0, a) := a,$$

$$J(m+1, a) := \{s \in S_{\omega^\alpha}^u \mid \forall_r (\forall_{t <_r} t \in J(m, a) \rightarrow \forall_{t <_{r+\Omega^s}} t \in J(m, a))\}.$$

Let us verify an auxiliary result that we will need later, namely the implication

$$\text{Prog}(J(m, a)) \rightarrow \text{Prog}(J(m+1, a)).$$

Aiming at $\text{Prog}(J(m+1, a))$ we fix $s \in \varepsilon(S_{\omega^\alpha}^u)$ and assume $\forall_{t <_s} t \in J(m+1, a)$. Our goal is to establish $s \in J(m+1, a)$, which is equivalent to

$$\forall_r (\forall_{t <_r} t \in J(m, a) \rightarrow \forall_{t <_{r+\Omega^s}} t \in J(m, a)).$$

If $s = 0$ this is easy: From $\forall_{t <_r} t \in J(m, a)$ and the assumption $\text{Prog}(J(m, a))$ we get $r \in J(m, a)$, so that we have $\forall_{t <_{r+\Omega^0}} t \in J(m, a)$. In case $s > 0$ any $t <_{r+\Omega^s}$ is smaller than some term $r + \Omega^{s_0} \cdot \beta$, with $s_0 < s$ and $\beta < \omega^\alpha$. We establish $\forall_{t <_{r+\Omega^{s_0} \cdot \beta}} t \in J(m, a)$ by induction on β . For $\beta = 0$ it suffices to cite the assumption $\forall_{t <_r} t \in J(m, a)$. If β is a limit ordinal then any $t <_{r+\Omega^{s_0} \cdot \beta}$ is smaller than $r + \Omega^{s_0} \cdot \beta_0$ for some $\beta_0 < \beta$, and the induction step is immediate. Now assume that $\beta = \beta_0 + 1$ is a successor. As $s_0 < s$, one of the assumptions above provides $s_0 \in J(m+1, a)$, which implies

$$\forall_{t <_{r+\Omega^{s_0} \cdot \beta_0}} t \in J(m, a) \rightarrow \forall_{t <_{r+\Omega^{s_0} \cdot \beta_0 + \Omega^{s_0}}} t \in J(m, a).$$

In view of $r + \Omega^{s_0} \cdot \beta_0 + \Omega^{s_0} = r + \Omega^{s_0} \cdot \beta$ this completes the induction step. After these preparations, let us prove

$$\forall_{m \in \omega} (\text{Prog}(J(m, a)) \rightarrow \forall_{t < \Omega_n^{\varepsilon_\sigma + 1}} t \in J(m, a))$$

by induction on (σ, n) . As observed above the instance $m = 0$ suffices to establish the lemma; the other instances are required to perform the induction, while keeping a fixed and the statement primitive recursive. In the induction step we assume $\text{Prog}(J(m, a))$ for some m and consider an arbitrary $t < \Omega_n^{\varepsilon_\sigma + 1}$. First assume $n = 0$, such that we have $\Omega_n^{\varepsilon_\sigma + 1} = \varepsilon_\sigma + 1$. If $t < \varepsilon_\sigma$ then we have $t < \Omega_k^{\varepsilon_\tau + 1}$ for some $\tau < S_{\omega^\alpha}^u \sigma$ and some $k \in \omega$. So $t \in J(m, a)$ holds by the induction hypothesis. Having shown this much, we can conclude $\varepsilon_\sigma \in J(m, a)$ by $\text{Prog}(J(m, a))$. Together we have established $t \in J(m, a)$ for all $t < \varepsilon_\sigma + 1 = \Omega_0^{\varepsilon_\sigma + 1}$, as required. Now assume $n > 1$ and write $n = k + 1$. Our auxiliary result provides $\text{Prog}(J(m + 1, a))$ and the induction hypothesis yields $\forall_{r < \Omega_k^{\varepsilon_\sigma + 1}} r \in J(m + 1, a)$. Using $\text{Prog}(J(m + 1, a))$ again we get $\Omega_k^{\varepsilon_\sigma + 1} \in J(m + 1, a)$, which is equivalent to

$$\forall_r (\forall_{t < r} t \in J(m, a) \rightarrow \forall_{t < r + \Omega_n^{\varepsilon_\sigma + 1}} t \in J(m, a)).$$

With $r = 0$ the antecedent is trivial and we get $\forall_{t < \Omega_n^{\varepsilon_\sigma + 1}} t \in J(m, a)$ as desired. \square

To obtain a well-ordering principle in the sense of Definition 2.1 we must also verify compatibility:

Lemma 2.12. *For $\alpha < \beta$ we have $\varepsilon(S_{\omega^\alpha}^u) \subseteq \varepsilon(S_{\omega^\beta}^u)$, and $<_{\varepsilon(S_{\omega^\alpha}^u)}$ is the restriction of $<_{\varepsilon(S_{\omega^\beta}^u)}$ to $\varepsilon(S_{\omega^\alpha}^u)$. Also, we have $\varepsilon(S_{\omega^\lambda}^u) = \bigcup_{\gamma < \lambda} \varepsilon(S_{\omega^\gamma}^u)$ for each limit λ .*

Proof. Recall the auxiliary set $\varepsilon^0(S_{\omega^\alpha}^u) \supseteq \varepsilon(S_{\omega^\alpha}^u)$ discussed just after Definition 2.8. For $s, t \in \varepsilon^0(S_{\omega^\alpha}^u)$ one verifies

$$s \in \varepsilon(S_{\omega^\alpha}^u) \Leftrightarrow s \in \varepsilon(S_{\omega^\beta}^u)$$

and

$$s <_{\varepsilon(S_{\omega^\alpha}^u)} t \Leftrightarrow s <_{\varepsilon(S_{\omega^\beta}^u)} t$$

by simultaneous induction on $|s|$ resp. $|s| + |t|$. The base of the induction relies on the fact that the search trees S_γ^u are compatible, as shown in Lemma 1.7. The induction step is straightforward. To save some work, observe that it suffices to establish the implication “ \Rightarrow ” in the second biconditional, as we already know that both orderings are linear. We have thus established

$$\varepsilon(S_{\omega^\alpha}^u) = \varepsilon(S_{\omega^\beta}^u) \cap \varepsilon^0(S_{\omega^\alpha}^u)$$

and $<_{\varepsilon(S_{\omega^\alpha}^u)} = <_{\varepsilon(S_{\omega^\beta}^u)} \cap (\varepsilon(S_{\omega^\alpha}^u) \times \varepsilon(S_{\omega^\alpha}^u))$. The remaining claim about limit ordinals is reduced to the inclusion

$$\varepsilon^0(S_{\omega^\lambda}^u) \subseteq \bigcup_{\gamma < \lambda} \varepsilon^0(S_{\omega^\gamma}^u),$$

which is readily verified by induction on the length $|t|$ of a term $t \in \varepsilon^0(S_{\omega^\lambda}^u)$. Again this relies on Lemma 1.7, i.e. the corresponding statement for search trees. \square

Finally, we need to construct a rank function:

Lemma 2.13. *There is a primitive recursive function $(u, s) \mapsto |s|_{\varepsilon(S_{\omega^\cdot})}^u$ such that we have*

$$|s|_{\varepsilon(S_{\omega^\cdot})}^u = \begin{cases} \min\{\alpha \in \text{Ord} \mid s \in \varepsilon(S_{\omega^{\alpha+1}}^u)\} & \text{if such an } \alpha \text{ exists,} \\ \{1\} & \text{otherwise.} \end{cases}$$

Proof. As in the previous proof we use the auxiliary sets $\varepsilon^0(S_{\omega^{\alpha+1}}^u) \supseteq \varepsilon(S_{\omega^{\alpha+1}}^u)$ introduced after Definition 2.8. These sets consist of simple terms, built from “constants” ε_σ and β with $\sigma \in S_{\omega^{\alpha+1}}^u$ and $1 < \beta < \omega^{\alpha+1}$, respectively. Clearly we can check whether a given set s represents such a simple term, and if it does we can extract the constants it contains. Next we must check whether these constants have the required form: Lemma 1.9 provides a primitive recursive rank function for the search trees. Given an alleged constant ε_σ we can thus determine whether σ lies in some search tree $S_{\gamma+1}^u$, and we can compute the minimal such γ if it does. Bounded minimization gives $\alpha = \min\{\beta \leq \gamma \mid \gamma < \omega^{\beta+1}\}$, which is minimal with $\sigma \in S_{\omega^{\alpha+1}}^u$. Doing this for all constants in the term s we get the minimal α with $s \in \varepsilon^0(S_{\omega^{\alpha+1}}^u)$, or we come to decide that no such α exists. Once α is computed we check whether s lies in $\varepsilon(S_{\omega^{\alpha+1}}^u) \subseteq \varepsilon^0(S_{\omega^{\alpha+1}}^u)$: If it does, set $|s|_{\varepsilon(S_{\omega^\cdot})}^u = \alpha$, otherwise $|s|_{\varepsilon(S_{\omega^\cdot})}^u = \{1\}$. \square

Combining these lemmata with the results of the previous section we obtain the following:

Proposition 2.14. *Consider a countable transitive set $u = \{u_i \mid i \in \omega\}$. If there is no admissible set \mathbb{A} with $u \subseteq \mathbb{A}$ then $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ is a well-ordering principle.*

Proof. The previous lemmata show that $(\varepsilon(S_{\omega^\alpha}^u), <_{\varepsilon(S_{\omega^\alpha}^u)})$ are compatible linear orderings with a rank function. Now assume that u is not contained in a transitive model of Kripke-Platek set theory. By Corollary 1.5 this implies that none of the search trees $S_{\omega^\alpha}^u$ has an infinite branch. Using Lemma 1.6 we conclude that $S_{\omega^\alpha}^u$ is well-ordered by $<_{S_{\omega^\alpha}^u}$. Lemma 2.11 tells us that $(\varepsilon(S_{\omega^\alpha}^u), <_{\varepsilon(S_{\omega^\alpha}^u)})$ is a well-ordering for each $\alpha > 1$. As $\varepsilon(S_{\omega^0}^u) \subseteq \varepsilon(S_{\omega^1}^u)$ the case $\alpha = 0$ is covered as well. \square

If $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ is a well-ordering principle then the higher Bachmann-Howard principle yields a collapsing function $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$, for some ordinal α . In the rest of this section we establish properties of such a collapse. To get started, observe that $0 \in \varepsilon(S_{\omega^\alpha}^u)$ implies $\vartheta(0) \in \alpha$ and thus $\alpha > 0$. So we have $1 < \omega^1 \leq \omega^\alpha$, which means that $\varepsilon(S_{\omega^\alpha}^u)$ contains the terms $1 := \Omega^0 \cdot 1$ and $\Omega := \Omega^1 \cdot 1$. We observe the following:

Lemma 2.15. *The map*

$$\beta \mapsto \hat{\beta} := \begin{cases} 0 & \text{if } \beta = 0, \\ \Omega^0 \cdot \beta & \text{otherwise} \end{cases}$$

is an order isomorphism between ω^α and $\varepsilon(S_{\omega^\alpha}^u) \cap \Omega := \{t \in \varepsilon(S_{\omega^\alpha}^u) \mid t <_{\varepsilon(S_{\omega^\alpha}^u)} \Omega\}$.

Proof. It is clear from the definition of $<_{\varepsilon(S_{\omega^\alpha}^u)}$ that $\beta \mapsto \hat{\beta}$ is an order embedding. Also, we can infer $\Omega^0 \cdot \beta <_{\varepsilon(S_{\omega^\alpha}^u)} \Omega$ from $0 <_{\varepsilon(S_{\omega^\alpha}^u)} 1$. It remains to check that any term $t <_{\varepsilon(S_{\omega^\alpha}^u)} \Omega$ is of the form $t = 0$ or $t = \Omega^0 \cdot \beta$. First, aiming at a contradiction, assume that t is of the form ε_σ . Then $t <_{\varepsilon(S_{\omega^\alpha}^u)} \Omega$ would imply $\varepsilon_\sigma <_{\varepsilon(S_{\omega^\alpha}^u)} 1$ and thus $\varepsilon_\sigma <_{\varepsilon(S_{\omega^\alpha}^u)} 0$, which is false. Now assume that $t <_{\varepsilon(S_{\omega^\alpha}^u)} \Omega$ is of the form

$$t = \Omega^{t_0} \cdot \gamma_0 + \dots + \Omega^{t_m} \cdot \gamma_m.$$

We cannot have $t_0 = 1$ as this would require $\gamma_0 < 1$, which was not allowed in our term system. Thus we must have $t_0 <_{\varepsilon(S_{\omega^\alpha}^u)} 1$, which is easily seen to imply $t_0 = 0$. The latter makes $t_1 <_{\varepsilon(S_{\omega^\alpha}^u)} t_0$ impossible, so that we see $m = 0$ and $t = \Omega^0 \cdot \gamma_0$. \square

As we will argue in the same context for quite a while it is worth introducing some abbreviations:

Notation 2.16. In view of the previous lemma we will write β instead of $\hat{\beta}$ and $<$ instead of $<_{\varepsilon(S_{\omega^\alpha}^u)}$. Concerning the rank functions $|\cdot|_{\mathbf{L}}^u$ and $|\cdot|_{\mathbf{S}}^u$ we will omit the sub- and superscript, writing $|a|$ resp. $|\sigma|$ instead of $|a|_{\mathbf{L}}^u$ resp. $|\sigma|_{\mathbf{S}}^u$. This is harmless because these two rank functions are closely connected: For example we have $|a|_{\mathbf{L}}^u = |\langle a \rangle|_{\mathbf{S}}^u$. The rank function $|\cdot|_{\varepsilon(S_{\omega^\cdot}^u)}$ behaves quite differently and must be carefully distinguished: To make this visual we write s^* at the place of $|s|_{\varepsilon(S_{\omega^\cdot}^u)}$ (this is inspired by Rathjen's ordinal notation system for the Bachmann-Howard ordinal in [RV15]). To get some intuition for the notation, let us consider two ways to view an ordinal $\beta < \omega^\alpha$: First, one can view β as an element of $\mathbf{L}_{\omega^\alpha}^u$, with

$$|\beta| = |\beta|_{\mathbf{L}}^u = \min\{\gamma \in \text{Ord} \mid \beta \in \mathbb{L}_{\gamma+1}^u\} \leq \beta.$$

Note that inequality is possible, e.g. if we have $\beta \in u$. Alternatively, one can view β as the term $\hat{\beta}$ in $\varepsilon(S_{\omega^\alpha}^u)$. Then we have

$$\beta^* = |\beta|_{\varepsilon(S_{\omega^\cdot}^u)} = \min\{\gamma \in \text{Ord} \mid \beta < \omega^{\gamma+1}\} \leq \beta,$$

which implies $\omega^{\beta^*} \leq \beta < \omega^{\beta^*+1}$ in case $\beta > 0$.

The reader may wonder why we consider the well-ordering principle $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ rather than $\alpha \mapsto \varepsilon(S_\alpha^u)$. The following two results should make this clear:

Lemma 2.17. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. Then we have $\beta \leq \vartheta(\beta)$ for each $\beta < \omega^\alpha$. In particular this implies $\alpha = \omega^\alpha$, i.e. the ordinal α must be an ε -number.

Proof. Clearly $\gamma < \beta < \omega^\alpha$ implies $\gamma^* \leq \beta^*$. By the definition of Bachmann-Howard collapse we conclude $\gamma^* \leq \beta^* < \vartheta(\beta)$ and then $\vartheta(\gamma) < \vartheta(\beta)$. Now $\beta \leq \vartheta(\beta)$ is established by the usual induction on β : We have

$$\vartheta(\beta) \geq \sup\{\vartheta(\gamma) + 1 \mid \gamma < \beta\} \geq \sup\{\gamma + 1 \mid \gamma < \beta\},$$

which implies the induction step in both successor and limit case. \square

Proposition 2.18. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For any $t \in \varepsilon(S_{\omega^\alpha}^u)$ with $\Omega \leq t$ the ordinal $\vartheta(t)$ is additively principal and bigger than ω .

Proof. From $\beta < \omega^{\beta^*+1}$ and $\gamma < \omega^{\gamma^*+1}$ we get $\beta + \gamma < \omega^{\max\{\beta^*, \gamma^*\}+1}$ and thus

$$(\beta + \gamma)^* \leq \max\{\beta^*, \gamma^*\}.$$

Also, $\beta < \omega^{\beta+1}$ yields $\beta^* \leq \beta$. Thus $\beta, \gamma < \vartheta(t)$ implies

$$(\beta + \gamma)^* \leq \max\{\beta^*, \gamma^*\} \leq \max\{\beta, \gamma\} < \vartheta(t).$$

As ω^α is additively principal we can form the term $\beta + \gamma = \Omega^0 \cdot (\beta + \gamma)$. From $\beta + \gamma < \Omega \leq t$ and the above we infer $\vartheta(\beta + \gamma) < \vartheta(t)$. Together with the previous lemma we obtain $\beta + \gamma < \vartheta(t)$, as needed to show that $\vartheta(t)$ is additively principal. As for $\omega < \vartheta(t)$, note first that we have $0^* = 0 \leq t^* < \vartheta(t)$ and thus $0 < \vartheta(0) < \vartheta(t)$, which means $1 < \vartheta(t)$. In view of $\omega^* = 1$ this implies $\vartheta(\omega) < \vartheta(t)$. Together with the previous lemma we get $\omega < \vartheta(t)$. \square

Note that the condition $\Omega \leq t$ in the statement of the proposition is necessary: We could indeed set $\vartheta(\beta) := f(\beta)$ for any strictly increasing function $f : \aleph_1 \rightarrow \aleph_1$ with $f(\beta) > \beta$, and then use the construction from Remark 2.6 to extend ϑ to a Bachmann-Howard collapse $\varepsilon(S_{\omega^{\aleph_1}}^u) \rightarrow \aleph_1$. The value $\vartheta(\Omega)$ cannot be prescribed in the same way because a strictly increasing function $f : \aleph_1 + 1 \rightarrow \aleph_1$ does not exist. It follows that the values $\vartheta(\beta)$ for $\beta < \Omega$ are not informative at all, in contrast to the usual notation systems for the Bachmann-Howard ordinal. This will not be a problem, however, as we can work with the ordinals $\vartheta(\Omega^{1+\beta})$ instead. Next, recall the above definition of exponentiation and addition for terms in $\varepsilon(S_{\omega^\alpha}^u)$. Crucially, these operations behave nicely with respect to the ranks:

Lemma 2.19. *The following holds for all $s, t \in \varepsilon(S_{\omega^\alpha}^u)$:*

- (i) $(\Omega^s)^* \leq s^*$,
- (ii) $(s + t)^* \leq \max\{s^*, t^*\}$.

Proof. The argument is essentially the same for both claims, so we only consider part (ii): Set $\delta := \max\{s^*, t^*\}$. By definition of the rank we have $s, t \in \varepsilon(S_{\omega^{\delta+1}}^u)$. We have already observed that $\varepsilon(S_{\omega^{\delta+1}}^u)$ is closed under addition, i.e. we also have $s + t \in \varepsilon(S_{\omega^{\delta+1}}^u)$. This implies $(s + t)^* \leq \delta$ by the minimality of the rank. \square

The following notions will be of central importance in the next sections, where we extend the search trees $S_{\omega^\alpha}^u$ to proof trees and analyse them by proof-theoretic methods:

Definition 2.20. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For each $t \in \varepsilon(S_{\omega^\alpha}^u)$ with $\Omega \leq t$ and each subset $X \subseteq \omega^\alpha$ we define a set $C^\vartheta(t, X) \subseteq \varepsilon(S_{\omega^\alpha}^u)$: Put

$$C^\vartheta(t, X) := \bigcup_{n \in \omega} C_n^\vartheta(t, X)$$

where $C_n^\vartheta(t, X)$ is inductively defined by

$$\begin{aligned} C_0^\vartheta(t, X) &= X \cup \{0\}, \\ C_{n+1}^\vartheta(t, X) &= C_n^\vartheta(t, X) \cup \{s \in \varepsilon(S_{\omega^\alpha}^u) \mid s^* \in C_n^\vartheta(t, X)\} \\ &\quad \cup \{\vartheta(s) \mid s \in C_n^\vartheta(t, X) \text{ and } s < t\} \\ &\quad \cup \{s \mid s < s' \text{ for some } s' \in C_n^\vartheta(t, X) \cap \Omega\}. \end{aligned}$$

Here we have abbreviated $C_n^\vartheta(t, X) \cap \Omega = \{s' \in C_n^\vartheta(t, X) \mid s' < \Omega\}$. The reader may wish to recall that s^* and $\vartheta(s)$ are elements of $\omega^\alpha \cong \varepsilon(S_{\omega^\alpha}^u) \cap \Omega$, for any term $s \in \varepsilon(S_{\omega^\alpha}^u)$. Clearly $C^\vartheta(t, X)$ is primitive recursive in ϑ, t, X, α and (the fixed enumeration of) u . Let us show some basic properties:

Lemma 2.21. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. Then the following holds:*

- (i) If $\Omega \leq t < t'$ then $C^\vartheta(t, X) \subseteq C^\vartheta(t', X)$.
- (ii) If $X \subseteq C^\vartheta(t, X') \cap \Omega$ then $C^\vartheta(t, X) \subseteq C(t, X')$.
- (iii) If $s \in C^\vartheta(t, X)$ then $s^* \in C^\vartheta(t, X)$.

Proof. (i) It is straightforward to establish $C_n^\vartheta(t, X) \subseteq C_n^\vartheta(t', X)$ by induction on n . (ii) Note that $C^\vartheta(t, X)$ is defined because of $X \subseteq \varepsilon(S_{\omega^\alpha}^u) \cap \Omega \cong \omega^\alpha$. A straightforward induction on n shows $C_n^\vartheta(t, X) \subseteq C(t, X')$. (iii) Let us first establish the claim in the special case $s < \Omega$: In the context of Notation 2.16 we have observed $s^* \leq s$. Thus $s \in C^\vartheta(t, X) \cap \Omega$ implies $s^* \in C^\vartheta(t, X)$,

by one of the closure properties of $C^\vartheta(t, X)$. As for the general case, let us show by induction on n that $s \in C_n^\vartheta(t, X)$ implies $s^* \in C^\vartheta(t, X)$: For $n = 0$ this reduces to the special case $s < \Omega$, because of $X \cup \{0\} \subseteq C^\vartheta(t, X) \cap \Omega$. Concerning the induction step, if $s \in C_{n+1}^\vartheta(t, X)$ holds because of $s^* \in C_n^\vartheta(t, X)$ then the claim is immediate. In all other cases we have $s < \Omega$, and the claim holds as before. \square

The following recovers the usual construction of the Bachmann-Howard ordinal (cf. Remark 2.6 above):

Lemma 2.22. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For each $t \in \varepsilon(S_{\omega^\alpha}^u)$ with $\Omega \leq t$ we have*

$$C^\vartheta(t, \vartheta(t)) \cap \Omega = \vartheta(t).$$

Proof. First, $\vartheta(t) \subseteq C^\vartheta(t, \vartheta(t)) \cap \Omega$ is immediate by the definition of $C_0^\vartheta(t, \vartheta(t))$. To establish the converse inclusion we prove

$$C_n^\vartheta(t, \vartheta(t)) \cap \Omega \subseteq \vartheta(t)$$

by induction on n . Simultaneously we show that $s \in C_n^\vartheta(t, \vartheta(t))$ implies $s^* < \vartheta(t)$. For $n = 0$ the inclusion $\vartheta(t) \cup \{0\} \subseteq \vartheta(t)$ amounts to $0 < \vartheta(t)$, which holds by Proposition 2.18. Also $s < \vartheta(t)$ implies $s^* \leq s < \vartheta(t)$, as in the proof of the previous lemma. In the induction step we distinguish several cases: First, assume that $s \in C_{n+1}^\vartheta(t, \vartheta(t))$ holds because of $s^* \in C_n^\vartheta(t, \vartheta(t))$. In view of $s^* < \Omega$ the induction hypothesis yields $s^* < \vartheta(t)$. In case $s < \Omega$ we also have $s < \Omega \leq t$. Then $\vartheta(s) < \vartheta(t)$ follows by the definition of Bachmann-Howard collapse. Together with Lemma 2.17 we obtain $s \leq \vartheta(s) < \vartheta(t)$, as required for the induction step. Next, assume $s = \vartheta(s_0)$ with $s_0 \in C_n^\vartheta(t, X)$ and $s_0 < t$. The induction hypothesis provides $s_0^* < \vartheta(t)$, which implies $s = \vartheta(s_0) < \vartheta(t)$. Also, because of $s = \vartheta(s_0) < \Omega$ we have $s^* \leq s < \vartheta(t)$. Finally, assume that we have $s < s'$ for some $s' \in C_n^\vartheta(t, X) \cap \Omega$. By induction hypothesis we have $s' < \vartheta(t)$. This implies $s^* \leq s < s' < \vartheta(t)$, as required. \square

For a proof-theoretic analysis of the Kripke-Platek axioms it is convenient to use the following “controlling operators”. This formalism is due to Buchholz [Buc93]:

Definition 2.23. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For a term $t \in \varepsilon(S_{\omega^\alpha}^u)$ with $\Omega \leq t$ and ordinals $\beta_1, \dots, \beta_k < \omega^\alpha$ we write

$$\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] := C^\vartheta(t + 1, \{\beta_1, \dots, \beta_k\})$$

We will write \mathcal{H}_t^ϑ rather than $\mathcal{H}_t^\vartheta[]$ (i.e. in case $k = 0$).

Occasionally it is helpful to think of the operator $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ as the function

$$\gamma \mapsto \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k, \gamma],$$

which maps an ordinal $\gamma < \omega^\alpha$ to a subset of $\varepsilon(S_{\omega^\alpha}^u)$. Note that this function exists as a set, as it is the restriction of a primitive recursive class function to a set. Our next goal is to recover properties of Buchholz’ operators in our context:

Lemma 2.24. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. The operators $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ are “closure operators”, in the sense that we have*

- (i) $\{\beta_1, \dots, \beta_k\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$,
- (ii) if $\{\gamma_1, \dots, \gamma_l\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ then $\mathcal{H}_t^\vartheta[\gamma_1, \dots, \gamma_l] \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$.

Proof. Part (i) is immediate by definition, and (ii) holds by Lemma 2.21(ii). \square

Let us record an easy consequence:

Lemma 2.25. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. Then the following holds:*

- (i) *If $s \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] \cap \Omega$ and $s' < s$ then $s' \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$,*
- (ii) *if $\max\{\gamma_1, \dots, \gamma_l\} \leq \max\{\beta_1, \dots, \beta_k\}$ then $\mathcal{H}_t^\vartheta[\gamma_1, \dots, \gamma_l] \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$.*

Proof. Part (i) holds by the definition of $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] = C^\vartheta(t+1, \{\beta_1, \dots, \beta_k\})$. As for (ii), the previous lemma implies $\max\{\beta_1, \dots, \beta_k\} \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$. Then $\max\{\gamma_1, \dots, \gamma_l\} \leq \max\{\beta_1, \dots, \beta_k\}$ implies $\{\gamma_1, \dots, \gamma_l\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$, by (i). The previous lemma yields $\mathcal{H}_t^\vartheta[\gamma_1, \dots, \gamma_l] \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$, as desired. \square

Lemma 2.26. *The operators $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ are “nice”, in the sense that we have*

- (i) $\{0, \omega\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$,
- (ii) *if $\{s, s'\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ then $\{\Omega^s, s + s'\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$,*
- (iii) *if $\sigma \in S_{\omega^\alpha}^u$ and $|\sigma| \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ then $\varepsilon_\sigma \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$.*

Proof. (i) By the definition of $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] = C^\vartheta(t+1, \{\beta_1, \dots, \beta_k\})$ we immediately get $0 \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$. Now $1 < \omega^{0+1}$ implies $1^* = 0$, so that we obtain $1 \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$. Similarly, $\omega^1 \leq \omega < \omega^2$ implies $\omega^* = 1$, from which we can infer $\omega \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$.

(ii) Assume that $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ contains s and s' . By Lemma 2.21 it contains s^* and $(s')^*$. Using Lemma 2.19 and the previous lemma we infer that it contains $(\Omega^s)^*$ and $(s + t)^*$. Finally, $\{\Omega^s, s + t\} \subseteq \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ follows by definition.

(iii) By definition of the rank we have $\sigma \in S_{|\sigma|+1}^u \subseteq S_{\omega^{|\sigma|+1}}^u$ and thus $\varepsilon_\sigma \in \varepsilon(S_{\omega^{|\sigma|+1}}^u)$. This means $\varepsilon_\sigma^* \leq |\sigma|$. Thus $|\sigma| \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ implies $\varepsilon_\sigma^* \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$, and then $\varepsilon_\sigma \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ as desired. \square

The above are general conditions for “reasonable” operators. Now we show a result that is specific to the operators $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$:

Proposition 2.27. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. The operators $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ have the following properties:*

- (i) *If $\Omega \leq t < t'$ then $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] \subseteq \mathcal{H}_{t'}^\vartheta[\beta_1, \dots, \beta_k]$,*
- (ii) *if $s \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$ and $s \leq t$ then $\vartheta(s) \in \mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k]$,*
- (iii) *if $s \in \mathcal{H}_t^\vartheta \cap \Omega$ then $s < \vartheta(t+1)$,*
- (iv) *if $\{s, t\} \subseteq \mathcal{H}_t^\vartheta$ and $s < s'$ then $\vartheta(t + \Omega^s) < \vartheta(t + \Omega^{s'})$.*

Proof. (i) This is a special case of Lemma 2.21.

(ii) Immediate by the closure properties of $\mathcal{H}_t^\vartheta[\beta_1, \dots, \beta_k] = C^\vartheta(t+1, \beta_1, \dots, \beta_k)$.

(iii) Using Lemma 2.21(ii) we get

$$\mathcal{H}_t^\vartheta = C^\vartheta(t+1, \emptyset) \subseteq C^\vartheta(t+1, \vartheta(t+1)).$$

Then the claim follows from Lemma 2.22.

(iv) From $s < s'$ we get $t + \Omega^s < t + \Omega^{s'}$. To infer $\vartheta(t + \Omega^s) < \vartheta(t + \Omega^{s'})$ it remains to establish $(t + \Omega^s)^* < \vartheta(t + \Omega^{s'})$. Now $\{s, t\} \subseteq \mathcal{H}_t^\vartheta$ implies $t + \Omega^s \in \mathcal{H}_t^\vartheta = C^\vartheta(t+1, \emptyset)$. Using Lemma 2.21 we get

$$(t + \Omega^s)^* \in C^\vartheta(t+1, \emptyset) \subseteq C^\vartheta(t + \Omega^{s'}, \emptyset) \subseteq C^\vartheta(t + \Omega^{s'}, \vartheta(t + \Omega^{s'})).$$

Together with $(t + \Omega^s)^* < \Omega$ Lemma 2.22 yields $(t + \Omega^s)^* < \vartheta(t + \Omega^{s'})$, as needed. \square

3. FROM SEARCH TREE TO PROOF TREE

In Section 1 we have built a “search tree” S_α^u for each countable transitive set $u = \{u_i \mid i \in \omega\}$ and each ordinal α . As stated there, S_α^u can be seen as an attempted proof of a contradiction in \mathbb{L}_α^u -logic, with the axioms of Kripke-Platek set theory as open assumptions. The goal of this section is to remove these assumptions, by adding infinitary proofs of the Kripke-Platek axioms. To begin, we give a reasonably general definition of infinitary proof trees, which we call $\mathbf{L}_{\omega^\alpha}^u$ -preproofs.

Recall that the search tree $S_{\omega^\alpha}^u$ is a subtree of $(\mathbf{L}_{\omega^\alpha}^u)^{<\omega}$, where each node is labelled by an $\mathbf{L}_{\omega^\alpha}^u$ -sequent. The order of formulas in a sequent was crucial for the definition of the search trees, but it is inessential in the context of proof trees. We will thus identify a sequent with the set of its entries (e.g. $\Gamma \subseteq \Delta$ expresses that each entry of Γ is also an entry of Δ). In addition to sequents, $\mathbf{L}_{\omega^\alpha}^u$ -preproofs will carry labels for “rules”: By an $\mathbf{L}_{\omega^\alpha}^u$ -rule we shall mean a symbol from the list

$$\begin{aligned} \text{Ax}, \quad (\wedge, \psi_0, \psi_1), \quad (\vee_i, \psi_0, \psi_1), \quad (\forall_x, \psi), \quad (\exists_x, a, \psi), \\ (\text{Cut}, \psi), \quad (\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta), \quad (\text{Rep}, a), \end{aligned}$$

where ψ_0, ψ_1 and ψ are $\mathbf{L}_{\omega^\alpha}^u$ -formulas; we have $i \in \{0, 1\}$; a is an element of $\mathbf{L}_{\omega^\alpha}^u$; and θ is a bounded disjunction which does not contain the variable z . In addition to the rules, each node of an $\mathbf{L}_{\omega^\alpha}^u$ -preproof will be labelled by an element of the term system $\varepsilon(S_{\omega^\alpha}^u)$, defined in the previous section. We do not assume that $\varepsilon(S_{\omega^\alpha}^u)$ is well-founded. For this reason the term “preproof” is better than “proof”, even though we will occasionally use the latter for the sake of brevity; also, we will sometimes refer to the elements of $\varepsilon(S_{\omega^\alpha}^u)$ as “ordinal labels”. As in the previous section we will write $<$ rather than $<_{\varepsilon(S_{\omega^\alpha}^u)}$ for the order relation on $\varepsilon(S_{\omega^\alpha}^u)$ (recall from Lemma 2.15 that ω^α can be identified with an initial segment of this ordering).

Definition 3.1. Consider a countable transitive set $u = \{u_i \mid i \in \omega\} \supseteq \{0, 1\}$ and an ordinal $\alpha > 1$. An $\mathbf{L}_{\omega^\alpha}^u$ -preproof consists of a non-empty tree $P \subseteq (\mathbf{L}_{\omega^\alpha}^u)^{<\omega}$ and labelling functions $l : P \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-sequents”}$, $r : P \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-rules”}$ and $o : P \rightarrow \varepsilon(S_{\omega^\alpha}^u)$ such that the following “local correctness conditions” hold at every node $\sigma \in P$:

If $r(\sigma)$ is then ...
Ax	σ is a leaf of P and $l(\sigma)$ contains a true Δ_0 -formula,
(\wedge, ψ_0, ψ_1)	we have $\sigma \frown a \in P$ iff $a \in \{0, 1\}$, and $o(\sigma \frown a) < o(\sigma)$; also $\psi_0 \wedge \psi_1 \in l(\sigma)$ and $l(\sigma \frown i) \subseteq l(\sigma), \psi_i$ for $i = 0, 1$,
(\vee_i, ψ_0, ψ_1)	we have $\sigma \frown a \in P$ iff $a = 0$, and $o(\sigma \frown 0) < o(\sigma)$; also $\psi_0 \vee \psi_1 \in l(\sigma)$ and $l(\sigma \frown 0) \subseteq l(\sigma), \psi_i$,
(\forall_x, ψ)	we have $\sigma \frown a \in P$ for all $a \in \mathbf{L}_{\omega^\alpha}^u$, and $o(\sigma \frown a) + \omega \leq o(\sigma)$; also $\forall_x \psi \in l(\sigma)$ and $l(\sigma \frown a) \subseteq l(\sigma), \psi(a)$ for each a ,
(\exists_x, b, ψ)	we have $\sigma \frown a \in P$ iff $a = 0$, and $o(\sigma \frown 0) + \omega \leq o(\sigma)$ and $ b < o(\sigma)$; also $\exists_x \psi \in l(\sigma)$ and $l(\sigma \frown 0) \subseteq l(\sigma), \psi(b)$,

(Cut, ψ)	we have $\sigma \frown a \in P$ iff $a \in \{0, 1\}$, and $o(\sigma \frown a) < o(\sigma)$; also $l(\sigma \frown 0) \subseteq l(\sigma)$, ψ and $l(\sigma \frown 1) \subseteq l(\sigma)$, $\neg\psi$,
(Ref, $\exists_z \forall_{x \in a} \exists_{y \in z} \theta$)	we have $\sigma \frown a \in P$ iff $a = 0$, and $o(\sigma \frown 0) < o(\sigma)$ and $\Omega \leq o(\sigma)$; also $\exists_z \forall_{x \in a} \exists_{y \in z} \theta \in l(\sigma)$ and $l(\sigma \frown 0) \subseteq l(\sigma)$, $\forall_{x \in a} \exists_y \theta$,
(Rep, b)	we have $\sigma \frown a \in P$ iff $a = b$, and $o(\sigma \frown b) < o(\sigma)$; also $l(\sigma \frown b) \subseteq l(\sigma)$.

We call $l(\langle \rangle)$, $r(\langle \rangle)$ and $o(\langle \rangle)$ the end-sequent, the last rule, and the height of the preproof P , respectively.

Let us give $\mathbf{L}_{\omega^\alpha}^u$ -preproofs of the Kripke-Platek axioms:

Lemma 3.2. *Consider a countable transitive u and an ordinal $\alpha > 1$. Except for foundation, each Kripke-Platek axiom has an $\mathbf{L}_{\omega^\alpha}^u$ -preproof with height below $\Omega^1 \cdot 2$.*

Proof. Let us start with the case of Δ_0 -separation, i.e. an axiom of the form

$$\forall_{v_1} \cdots \forall_{v_k} \forall_x \exists_y (\forall_{z \in y} (z \in x \wedge \theta(x, z, \vec{v})) \wedge \forall_{z \in x} (\theta(x, z, \vec{v}) \rightarrow z \in y))$$

with a Δ_0 -formula θ . The nodes in the desired $\mathbf{L}_{\omega^\alpha}^u$ -preproof will be precisely those of the form $\langle c_1, \dots, c_k, a, 0 \rangle$, where $\vec{c}, a \in \mathbf{L}_{\omega^\alpha}^u$ are arbitrary. Given such parameters, set $\gamma := \max\{|c_1|, \dots, |c_k|, |a|\} + 1$ and observe that $\vec{c}, a \in \mathbf{L}_\gamma^u$ holds by definition of the rank. We can primitive recursively compute the set

$$b := \{z \in a \mid \theta(a, z, \vec{c})\} = \{z \in \mathbb{L}_\gamma \mid \mathbb{L}_\gamma \models \theta(a, z, \vec{c})\} \in \mathbb{L}_{\gamma+\omega}^u$$

and the ordinal

$$|b| := \min\{\beta < \gamma + \omega \mid b \in \mathbb{L}_{\beta+1}^u\} < \omega^\alpha,$$

which allows us to view b as an element of $\mathbf{L}_{\omega^\alpha}^u$. By construction of b the bounded $\mathbf{L}_{\omega^\alpha}^u$ -formula

$$\forall_{z \in b} (z \in a \wedge \theta(a, z, \vec{c})) \wedge \forall_{z \in a} (\theta(a, z, \vec{c}) \rightarrow z \in b)$$

is true. Thus the leaf $\langle c_1, \dots, c_k, a, 0 \rangle$ can be labelled by

$$l(\langle c_1, \dots, c_k, a, 0 \rangle) = \langle \forall_{z \in b} (z \in a \wedge \theta(a, z, \vec{c})) \wedge \forall_{z \in a} (\theta(a, z, \vec{c}) \rightarrow z \in b) \rangle,$$

$$r(\langle c_1, \dots, c_k, a, 0 \rangle) = \text{Ax},$$

$$o(\langle c_1, \dots, c_k, a, 0 \rangle) = 0,$$

in a locally correct way. Next, we can take b as a witness for an existential quantifier. This amounts to setting

$$l(\langle c_1, \dots, c_k, a \rangle) = \langle \exists_y (\forall_{z \in y} (z \in a \wedge \theta(a, z, \vec{c})) \wedge \forall_{z \in a} (\theta(a, z, \vec{c}) \rightarrow z \in y)) \rangle,$$

$$r(\langle c_1, \dots, c_k, a \rangle) = (\exists_y, b, \forall_{z \in y} (z \in a \wedge \theta(a, z, \vec{c})) \wedge \forall_{z \in a} (\theta(a, z, \vec{c}) \rightarrow z \in y)),$$

$$o(\langle c_1, \dots, c_k, a \rangle) = \Omega.$$

Concerning local correctness, note that we have $|b| < \Omega$ in the term system $\varepsilon(S_{\omega^\alpha}^u)$, as $|b|$ is an ordinal below ω^α (cf. Lemma 2.15). The above construction was performed for all values $a \in \mathbf{L}_{\omega^\alpha}^u$. Thus we may introduce a universal quantifier, by setting

$$l(\langle c_1, \dots, c_k \rangle) = \langle \forall_x \exists_y (\forall_{z \in y} (z \in x \wedge \theta(x, z, \vec{c})) \wedge \forall_{z \in x} (\theta(x, z, \vec{c}) \rightarrow z \in y)) \rangle,$$

$$r(\langle c_1, \dots, c_k \rangle) = (\forall_x, \exists_y (\forall_{z \in y} (z \in x \wedge \theta(x, z, \vec{c})) \wedge \forall_{z \in x} (\theta(x, z, \vec{c}) \rightarrow z \in y))),$$

$$o(\langle c_1, \dots, c_k \rangle) = \Omega + \omega.$$

The universal quantifiers over the variables v_i are introduced in the same way, increasing the height by ω for each quantifier. The root will then receive labels

$$\begin{aligned} l(\langle \rangle) &= \langle \forall \vec{v} \forall x \exists y (\forall z \in y (z \in x \wedge \theta(x, z, \vec{v})) \wedge \forall z \in x (\theta(x, z, \vec{v}) \rightarrow z \in y)) \rangle, \\ o(\langle \rangle) &= \Omega + \omega \cdot (k + 1), \end{aligned}$$

as required.

The axioms of equality, extensionality, pairing, union and infinity are proved in a similar way. For infinity one uses the witness $\omega \in \mathbf{L}_{\omega^\alpha}^u$ with $|\omega| \leq \omega$ (inequality is possible, e.g. if $\omega \in u$).

Let us now look at an instance of Δ_0 -collection, i.e. at an axiom

$$\forall \vec{v} \forall w (\forall x \in w \exists y \theta(x, y, \vec{v}, w) \rightarrow \exists z \forall x \in w \exists y \in z \theta(x, y, \vec{v}, w)),$$

where θ is a Δ_0 -formula. For arbitrary parameters $a, b, \vec{c}, d \in \mathbf{L}_{\omega^\alpha}^u$ the sequent

$$\neg \theta(a, b, \vec{c}, d), \theta(a, b, \vec{c}, d)$$

contains a true bounded formula, i.e. it is an axiom. Introducing an existential quantifier we get

$$\neg \theta(a, b, \vec{c}, d), \exists y \theta(a, y, \vec{c}, d)$$

with height Ω (as above, the rank of the witness b is always bounded by Ω). Since $b \in \mathbf{L}_{\omega^\alpha}^u$ was arbitrary the constructed preproofs can be combined, to obtain

$$\forall y \neg \theta(a, y, \vec{c}, d), \exists y \theta(a, y, \vec{c}, d)$$

with height $\Omega + \omega$. Together with the axiom $a \notin d, a \in d$ we get

$$a \in d \wedge \forall y \neg \theta(a, y, \vec{c}, d), a \notin d, \exists y \theta(a, y, \vec{c}, d)$$

with height $\Omega + \omega + 1$, and then

$$\exists x \in d \forall y \neg \theta(x, y, \vec{c}, d), a \notin d, \exists y \theta(a, y, \vec{c}, d)$$

with height $\Omega + \omega \cdot 2$. Introducing two disjunctions yields

$$\exists x \in d \forall y \neg \theta(x, y, \vec{c}, d), a \notin d \vee \exists y \theta(a, y, \vec{c}, d),$$

now with height $\Omega + \omega \cdot 2 + 2$. Since a was arbitrary this gives

$$\exists x \in d \forall y \neg \theta(x, y, \vec{c}, d), \forall x \in d \exists y \theta(x, y, \vec{c}, d)$$

with height $\Omega + \omega \cdot 3$. Now we are in a position to use the reflection rule

$$(\text{Ref}, \exists z \forall x \in d \exists y \in z \theta(x, y, \vec{c}, d)),$$

which gives a proof of

$$\exists x \in d \forall y \neg \theta(x, y, \vec{c}, d), \exists z \forall x \in d \exists y \in z \theta(x, y, \vec{c}, d)$$

with height $\Omega + \omega \cdot 3 + 1$. Observe that we have $\Omega < \Omega + \omega \cdot 3 + 1$, as required for the local correctness of the reflection rule. Introducing two disjunctions we obtain

$$\forall x \in d \exists y \theta(x, y, \vec{c}, d) \rightarrow \exists z \forall x \in d \exists y \in z \theta(x, y, \vec{c}, d)$$

with height $\Omega + \omega \cdot 3 + 3$. It only remains to introduce the universal quantifiers over w and \vec{v} , as in the case of Δ_0 -separation. \square

Now that we have proofs of the Kripke-Platek axioms, let us extend the search trees $S_{\omega^\alpha}^u$ from Section 1 to $\mathbf{L}_{\omega^\alpha}^u$ -preproofs of the empty sequent:

Proposition 3.3. *For each countable transitive set $u = \{u_i \mid i \in \omega\}$ and each ordinal $\alpha > 1$ there is an $\mathbf{L}_{\omega^\alpha}^u$ -preproof $(P_\alpha^u, l_\alpha^u, r_\alpha^u, o_\alpha^u)$ with end-sequent $\langle \rangle$ and height $\varepsilon_\langle \rangle \in \varepsilon(S_{\omega^\alpha}^u)$.*

Proof. We go through the construction of the search tree $S_{\omega^\alpha}^u$ with labelling function $l_\alpha^u = l : S_{\omega^\alpha}^u \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-sequents”}$ from Definition 1.1. In doing so we construct additional labellings $r_\alpha^u : S_{\omega^\alpha}^u \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-rules”}$ and $o_\alpha^u : S_{\omega^\alpha}^u \rightarrow \varepsilon(S_{\omega^\alpha}^u)$. We will also add certain subtrees to fulfil the local correctness conditions. Labelling the nodes in $S_{\omega^\alpha}^u$ by “ordinals” is easy: To the node $\sigma \in S_{\omega^\alpha}^u$ we attach the label $o_\alpha^u(\sigma) := \varepsilon_\sigma \in \varepsilon(S_{\omega^\alpha}^u)$. For $\sigma \smallfrown a \in S_{\omega^\alpha}^u$ we have $\sigma \smallfrown a <_{S_{\omega^\alpha}^u} \sigma$ in the Kleene-Brouwer ordering, and thus

$$\varepsilon_{\sigma \smallfrown a} < \varepsilon_{\sigma \smallfrown a} + \omega < \varepsilon_\sigma$$

by definition of the term system $\varepsilon(S_{\omega^\alpha}^u)$. This means that the labels $o_\alpha^u(\sigma)$ descend as required by the local correctness conditions. Next, for $\sigma \in S_{\omega^\alpha}^u$ we define the $\mathbf{L}_{\omega^\alpha}^u$ -rule $r_\alpha^u(\sigma)$ as follows: If σ has even length $2k$ then we put

$$r_\alpha^u(\sigma) = (\text{Cut}, \neg\theta_k).$$

Recall that $\langle \theta_k \rangle_{k \in \omega}$ is an enumeration of the Kripke-Platek axioms (excluding foundation) that we have used in the construction of the search trees. By definition of the search tree we have

$$l_\alpha^u(\sigma \smallfrown 0) \subseteq l_\alpha^u(\sigma), \neg\theta_k,$$

i.e. this part of the local correctness condition is satisfied. On the other hand, the search tree $S_{\omega^\alpha}^u$ does not contain the node $\sigma \smallfrown 1$. Here we must add a subtree to restore local correctness: In the previous lemma we have constructed an $\mathbf{L}_{\omega^\alpha}^u$ -preproof of θ_k , with ordinal height at most $\Omega^1 \cdot 2$. In view of $\Omega^1 \cdot 2 < \varepsilon_\sigma$ the monotonicity of the “ordinal” labelling is preserved when we attach this preproof to the node $\sigma \smallfrown 1$. Now consider a node $\sigma \in S_{\omega^\alpha}^u$ of odd length. We will only write out one case, leaving the remaining ones to the reader: Assume that we have

$$l_\alpha^u(\sigma) = \Gamma, \exists_x \psi(x), \Gamma'$$

where Γ consists of (negated) prime formulas. Recalling the definition of the search tree we see that $\sigma \smallfrown a \in S_{\omega^\alpha}^u$ holds precisely for $a = 0$. Also, neglecting the order of the formulas, we have

$$l_\alpha^u(\sigma \smallfrown 0) = l_\alpha^u(\sigma), \psi(b)$$

for some particular $b \in \mathbf{L}_{\omega^\alpha}^u$. As $|b|$ is an ordinal below ω^α we see

$$|b| < \Omega < \varepsilon_\sigma.$$

Setting

$$r_\alpha^u(\sigma) = (\exists_x, b, \psi)$$

we thus have local correctness at the node σ . □

To see where we would like to get, consider the following:

Proposition 3.4. *If an $\mathbf{L}_{\omega^\alpha}^u$ -preproof has ordinal height below Ω then some formula in its end-sequent holds in the structure $(\mathbb{L}_{\omega^\alpha}^u, \in)$. In particular the end-sequent cannot be empty.*

Proof. Writing (P, l, r, o) for the given $\mathbf{L}_{\omega^\alpha}^u$ -preproof, we have $o(\langle \rangle) < \Omega$ by assumption. By Lemma 2.15 this means that $o(\langle \rangle)$ is an actual ordinal (below ω^α). Thus it suffices to show

$$\forall \beta < \omega^\alpha \forall \sigma \in P (o(\sigma) = \beta \rightarrow \text{“some formula in } l(\sigma) \text{ holds in } \mathbb{L}_{\omega^\alpha}^u\text{”}).$$

This can be established by induction on β (note that the induction statement is primitive recursive): Consider some ordinal β and a node $\sigma \in P$ with $o(\sigma) = \beta$. We distinguish cases according to the rule $r(\sigma)$. First, observe that $r(\sigma)$ cannot be a reflection rule (Ref, ...), for this would require $\Omega \leq o(\sigma)$. The other cases follow from the local correctness conditions. As an example, consider $r(\sigma) = (\forall_x, \psi)$. Aiming at a contradiction, assume that no formula in $l(\sigma)$ holds in $\mathbb{L}_{\omega^\alpha}^u$. By local correctness the formula $\forall_x \psi(x)$ occurs in $l(\sigma)$. We want to establish that this formula holds in $\mathbb{L}_{\omega^\alpha}^u$. So consider an arbitrary element $a \in \mathbb{L}_{\omega^\alpha}^u$. Computing its rank

$$|a| = \min\{\beta < \omega^\alpha \mid a \in \mathbb{L}_{\beta+1}^u\}$$

we may view a as an element of $\mathbf{L}_{\omega^\alpha}^u$. By local correctness we have $\sigma \smallfrown a \in P$, as well as $o(\sigma \smallfrown a) < o(\sigma)$ and $l(\sigma \smallfrown a) \subseteq l(\sigma), \psi(a)$. The induction hypothesis tells us that some formula in $l(\sigma \smallfrown a)$ holds in $\mathbb{L}_{\omega^\alpha}^u$. We have assumed that all formulas in the sequent $l(\sigma)$ fail, which means that $\mathbb{L}_{\omega^\alpha}^u$ must satisfy $\psi(a)$. As $a \in \mathbb{L}_{\omega^\alpha}^u$ was arbitrary we can conclude that $\forall_x \psi$ holds in $\mathbb{L}_{\omega^\alpha}^u$, which completes the induction step in this case. It is also worth looking at a cut rule $r(\sigma) = (\text{Cut}, \psi)$. Let us assume that ψ holds in $\mathbb{L}_{\omega^\alpha}^u$, the converse case being symmetric. By local correctness we have $\sigma \smallfrown 1 \in P$, as well as $o(\sigma \smallfrown 1) < o(\sigma)$ and $l(\sigma \smallfrown 1) \subseteq l(\sigma), \neg\psi$. The induction hypothesis tells us that some formula in $l(\sigma \smallfrown 1)$ holds in $\mathbb{L}_{\omega^\alpha}^u$. By assumption the formula $\neg\psi$ fails in $\mathbb{L}_{\omega^\alpha}^u$. Thus $\mathbb{L}_{\omega^\alpha}^u$ must satisfy some formula in $l(\sigma)$, as required for the induction step. \square

In Proposition 3.3 we have constructed $\mathbf{L}_{\omega^\alpha}^u$ -preproofs P_α^u with empty end-sequent and ordinal height $\varepsilon_\langle \rangle > \Omega$. On the other hand we have just seen that no $\mathbf{L}_{\omega^\alpha}^u$ -preproof with empty end-sequent can have ordinal height below Ω . Now the plan is as follows: Aiming at a contradiction, assume that there is no admissible set that contains u . By Proposition 2.14 it follows that $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ is a well-ordering principle. The higher Bachmann-Howard principle from Definition 2.3 then gives a collapsing function $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$, for some ordinal α . This will allow us to “collapse” the $\mathbf{L}_{\omega^\alpha}^u$ -preproof P_α^u to ordinal height below Ω , yielding the desired contradiction. The required transformations of P_α^u use two techniques from proof-theory — cut elimination and collapsing — which will be developed in the next two sections. We point out that these methods stem from the ordinal analysis of Kripke-Platek set theory via local predicativity: see in particular the work of Jäger [Jäg82], Pohlers [Poh81] and Buchholz [Buc93].

4. CUT ELIMINATION

In the last paragraph of the previous section we have outlined an argument that establishes the existence of admissible sets. The present section is devoted to one particular step in this argument, cut elimination. We begin with the main concept:

Definition 4.1. The height $\text{ht}(\varphi) \in \omega$ of an $\mathbf{L}_{\omega^\alpha}^u$ -formula φ is defined as follows:

- (i) If φ is a Δ_0 -formula then we have $\text{ht}(\varphi) = 0$.

- (ii) If $\varphi \equiv \psi_0 \wedge \psi_1$ is not a Δ_0 -formula then $\text{ht}(\varphi) = \max\{\text{ht}(\psi_0), \text{ht}(\psi_1)\} + 1$. Similarly for $\varphi \equiv \psi_0 \vee \psi_1$.
- (iii) If $\varphi \equiv \forall_x \psi$ is not a Δ_0 -formula then we have $\text{ht}(\varphi) = \text{ht}(\psi) + 1$. Similarly for $\varphi \equiv \exists_x \psi$.

We say that an $\mathbf{L}_{\omega^\alpha}^u$ -preproof (P, l, r, o) has cut-rank n if the following holds: For any node $\sigma \in P$, if $r(\sigma)$ is of the form (Cut, ψ) then we have $\text{ht}(\psi) < n$.

Let us check the cut-rank of the proofs that we have constructed so far:

Lemma 4.2. *There is some global bound $C \in \omega$ such that the $\mathbf{L}_{\omega^\alpha}^u$ -preproofs P_α^u constructed in Proposition 3.3 all have cut-rank C .*

Proof. The $\mathbf{L}_{\omega^\alpha}^u$ -preproofs from Lemma 3.2, establishing the Kripke-Platek axioms, contain no cuts. Thus all cut rules in the preproofs P_α^u are of the form $(\text{Cut}, \neg\theta_k)$, where $\langle \theta_k \rangle_{k \in \omega}$ is our enumeration of the Kripke-Platek axioms (not including instances of foundation). As stated in Section 1 we only allow a fixed number C_0 of parameters in the axiom schemes of Δ_0 -separation and Δ_0 -collection (all other instances can be deduced via coding of tuples). Each such instance of Δ_0 -separation (resp. Δ_0 -collection) has height at most $C_0 + 2$ (resp. $C_0 + 5$). Thus the claim holds for $C = C_0 + 6$. \square

In the previous proof we have used the fact that the axioms in our search tree have bounded quantifier complexity. We should point out that this is a convenient simplification rather than a necessary prerequisite: It is indeed possible to transform a search tree with unbounded cut rank into a preproof with bounded cut rank. To do so one must interweave embedding and cut elimination, as in Rathjen's and Vizcaíno's construction of ω -models of bar induction (see [RV15, Theorem 5.26]). Rather than adopting this approach, we have chosen to bound the number of parameters in the axioms: It makes the presentation easier and means no real restriction (contrary to the situation for bar induction, where unbounded quantifier complexity cannot be avoided).

The goal of the present section is to transform the preproofs P_α^u into preproofs with cut-rank 2 (and unchanged end-sequent). The easiest way to describe the required operations would be by transfinite recursion over proof trees. However, this approach is not available to us: Firstly, we do not currently assume that the preproofs P_α^u are well-founded. More importantly, recursion over arbitrary well-orderings is not available in primitive recursive set theory. In order to describe the required operations in a primitive recursive way we adopt Buchholz' approach [Buc91] to "continuous cut elimination" (for systems of set theory this is worked out in [Buc01], but we will not follow the formalism from that paper). The idea is to define a set of $\mathbf{L}_{\omega^\alpha}^u$ -codes and a primitive recursive interpretation that maps each code to an $\mathbf{L}_{\omega^\alpha}^u$ -preproof. Primitive recursive transformations of proofs can then be described by simple operations on the codes. As a first step, let us define codes for the preproofs P_α^u that we have already constructed. To make the approach work we will also need codes $P_\alpha^u \sigma$ for subtrees of the preproofs P_α^u , rooted at arbitrary nodes $\sigma \in P_\alpha^u$.

Definition 4.3. By a basic $\mathbf{L}_{\omega^\alpha}^u$ -code we mean a term of the form $P_\alpha^u \sigma$, for a countable transitive set $u = \{u_i \mid i \in \omega\}$, an ordinal $\alpha > 1$, and a finite sequence σ with entries in $\mathbf{L}_{\omega^\alpha}^u$. We define functions

$$l_{\langle \rangle} : \text{"basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes"} \rightarrow \text{"}\mathbf{L}_{\omega^\alpha}^u\text{-sequents"},$$

$$\begin{aligned} r_{\langle \rangle} &: \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-rules”}, \\ o_{\langle \rangle} &: \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \rightarrow \varepsilon(S_{\omega^\alpha}^u) \end{aligned}$$

as follows: Let l_α^u , r_α^u and o_α^u be the labelling functions of the \mathbf{L}_α^u -preproof P_α^u (see Proposition 3.3). If σ is not a node in P_α^u then set $l_{\langle \rangle}(P_\alpha^u \sigma) = \langle 0 = 0 \rangle$, $r_{\langle \rangle}(P_\alpha^u \sigma) = \text{Ax}$ and $o_{\langle \rangle}(P_\alpha^u \sigma) = 0$. If σ is a node in P_α^u we put

$$\begin{aligned} l_{\langle \rangle}(P_\alpha^u \sigma) &= l_\alpha^u(\sigma), \\ r_{\langle \rangle}(P_\alpha^u \sigma) &= r_\alpha^u(\sigma), \\ o_{\langle \rangle}(P_\alpha^u \sigma) &= o_\alpha^u(\sigma). \end{aligned}$$

Also, we define a function

$$n : \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \times \mathbf{L}_{\omega^\alpha}^u \rightarrow \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”}, \quad n(P_\alpha^u \sigma, a) := P_\alpha^u(\sigma \frown a).$$

Intuitively, $P_\alpha^u \sigma$ represents the subtree of P_α^u rooted at σ . The functions $l_{\langle \rangle}$, $r_{\langle \rangle}$ and $o_{\langle \rangle}$ give the labels at the root of this subtree. Using n we can also access the children of a given node. Combining these functions allows us to recover the full tree P_α^u :

Definition 4.4. We iterate the function n from Definition 4.3 along finite sequences, to get a function

$$\bar{n} : \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \times (\mathbf{L}_{\omega^\alpha}^u)^{<\omega} \rightarrow \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”}$$

with

$$\begin{aligned} \bar{n}(P, \langle \rangle) &:= P, \\ \bar{n}(P, \sigma \frown a) &:= n(\bar{n}(P, \sigma), a). \end{aligned}$$

Using this, define functions

$$\begin{aligned} l &: \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \times (\mathbf{L}_{\omega^\alpha}^u)^{<\omega} \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-sequents”}, & l(P, \sigma) &:= l_{\langle \rangle}(\bar{n}(P, \sigma)), \\ r &: \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \times (\mathbf{L}_{\omega^\alpha}^u)^{<\omega} \rightarrow \text{“}\mathbf{L}_{\omega^\alpha}^u\text{-rules”}, & r(P, \sigma) &:= r_{\langle \rangle}(\bar{n}(P, \sigma)), \\ o &: \text{“basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes”} \times (\mathbf{L}_{\omega^\alpha}^u)^{<\omega} \rightarrow \varepsilon(S_{\omega^\alpha}^u), & o(P, \sigma) &:= o_{\langle \rangle}(\bar{n}(P, \sigma)). \end{aligned}$$

In order to define a subtree $[P] \subseteq (\mathbf{L}_{\omega^\alpha}^u)^{<\omega}$ for each basic $\mathbf{L}_{\omega^\alpha}^u$ -code P we need an auxiliary notion: The relevant premises of an $\mathbf{L}_{\omega^\alpha}^u$ -rule are given by

$$\begin{aligned} \iota(\text{Ax}) &= \emptyset, \\ \iota((\wedge, \psi_0, \psi_1)) &= \iota((\text{Cut}, \psi)) = \{0, 1\}, \\ \iota((\vee_i, \psi_0, \psi_1)) &= \iota((\exists_x, b, \psi)) = \iota((\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta)) = \{0\}, \\ \iota((\forall_x, \psi)) &= \mathbf{L}_{\omega^\alpha}^u, \\ \iota((\text{Rep}, a)) &= \{a\}. \end{aligned}$$

Given an $\mathbf{L}_{\omega^\alpha}^u$ -code P we now define $[P] \subseteq (\mathbf{L}_{\omega^\alpha}^u)^{<\omega}$ by recursion on the length of sequences: We always have $\langle \rangle \in [P]$. Given $\sigma \in [P]$ we stipulate $\sigma \frown a \in [P]$ if and only if $a \in \iota(r(P, \sigma))$. Finally, write $l_{[P]}$, $r_{[P]}$, $o_{[P]}$ for the restrictions of the functions $l(P, \cdot)$, $r(P, \cdot)$, $o(P, \cdot)$ to $[P]$. The tree $[P]$, together with the functions $l_{[P]}$, $r_{[P]}$, $o_{[P]}$, is called the interpretation of the basic $\mathbf{L}_{\omega^\alpha}^u$ -code P .

The following result is reassuring, even though we will never use it:

Lemma 4.5. *The labelled trees $[P_\alpha^u \langle \rangle]$ and P_α^u are equal. In particular $[P_\alpha^u \langle \rangle]$ is a locally correct $\mathbf{L}_{\omega^\alpha}^u$ -preproof.*

Proof. By induction on (the length of) a sequence $\sigma \in (\mathbf{L}_{\omega^\alpha}^u)^{<\omega}$ we verify

$$\sigma \in [P_\alpha^u] \Leftrightarrow \sigma \in P_\alpha^u.$$

Note that, once we have seen $\sigma \in [P_\alpha^u] \cap P_\alpha^u$, equality of the labels is immediate: Unravelling the definitions we get

$$l_{[P_\alpha^u]}(\sigma) = l(P_\alpha^u, \sigma) = l_\diamond(\bar{n}(P_\alpha^u, \sigma)) = l_\diamond(P_\alpha^u \sigma) = l_\alpha^u(\sigma),$$

where l_α^u is the labelling function of the proof P_α^u (the same holds for rules and ordinal labels). As for the base case of the induction, both $[P_\alpha^u]$ and P_α^u contain the empty sequent. Now assume that the equivalence holds for σ . By definition $[P_\alpha^u]$ and P_α^u are trees, so it suffices to consider the case where we have $\sigma \in [P_\alpha^u]$ and $\sigma \in P_\alpha^u$. We distinguish cases according to the rule $r_{[P_\alpha^u]}(\sigma) = r_\alpha^u(\sigma)$. Assume for example that $r_{[P_\alpha^u]}(\sigma)$ is of the form (\wedge, ψ_0, ψ_1) . By the definition of $[P_\alpha^u]$ we have $\sigma \frown a \in [P_\alpha^u]$ if and only if $a \in \{0, 1\}$. However, the latter is also equivalent to $\sigma \frown a \in P_\alpha^u$, by the local correctness of P_α^u . The other cases are checked in the same way. \square

We extend this result by showing that $[P]$ is an $\mathbf{L}_{\omega^\alpha}^u$ -preproof for any basic $\mathbf{L}_{\omega^\alpha}^u$ -code P . As we shall see, the proof of this is at least as important as the result.

Lemma 4.6. *The system of basic $\mathbf{L}_{\omega^\alpha}^u$ -codes is locally correct, in the sense that the following — which we will call condition (L) — holds for any basic $\mathbf{L}_{\omega^\alpha}^u$ -code P :*

<i>If $r_\diamond(P)$ is ...</i>	<i>... then ...</i>
Ax	$l_\diamond(P)$ contains a true Δ_0 -formula,
(\wedge, ψ_0, ψ_1)	we have $o_\diamond(n(P, i)) < o_\diamond(P)$ for $i = 0, 1$; also $\psi_0 \wedge \psi_1 \in l_\diamond(P)$ and $l_\diamond(n(P, i)) \subseteq l_\diamond(P), \psi_i$,
(\vee, ψ_0, ψ_1)	we have $o_\diamond(n(P, 0)) < o_\diamond(P)$; also $\psi_0 \vee \psi_1 \in l_\diamond(P)$ and $l_\diamond(n(P, 0)) \subseteq l_\diamond(P), \psi_i$,
(\forall_x, ψ)	we have $o_\diamond(n(P, a)) + \omega \leq o_\diamond(P)$ for all $a \in \mathbf{L}_{\omega^\alpha}^u$; also $\forall_x \psi \in l_\diamond(P)$ and $l_\diamond(n(P, a)) \subseteq l_\diamond(P), \psi(a)$,
(\exists_x, b, ψ)	we have $o_\diamond(n(P, 0)) + \omega \leq o_\diamond(P)$ and $ b < o_\diamond(P)$; also $\exists_x \psi \in l_\diamond(P)$ and $l_\diamond(n(P, 0)) \subseteq l_\diamond(P), \psi(b)$,
(Cut, ψ)	we have $o_\diamond(n(P, i)) < o_\diamond(P)$ for $i = 0, 1$; also $l_\diamond(n(P, 0)) \subseteq l_\diamond(P), \psi$ and $l_\diamond(n(P, 1)) \subseteq l_\diamond(P), \neg\psi$,
(Ref, $\exists_z \forall_{x \in a} \exists_{y \in z} \theta$)	we have $o_\diamond(n(P, 0)) < o_\diamond(P)$ and $\Omega \leq o_\diamond(P)$; also $\exists_z \forall_{x \in a} \exists_{y \in z} \theta \in l_\diamond(P)$ and $l_\diamond(n(P, 0)) \subseteq l_\diamond(P), \forall_{x \in a} \exists_{y \in z} \theta$,
(Rep, b)	we have $o_\diamond(n(P, b)) < o_\diamond(P)$; also $l_\diamond(n(P, b)) \subseteq l_\diamond(P)$.

Proof. Any basic $\mathbf{L}_{\omega^\alpha}^u$ -code is of the form $P = P_\alpha^u \sigma$. We must consider two cases: For $\sigma \notin P_\alpha^u$ we have defined $r_\diamond(P_\alpha^u \sigma) = \text{Ax}$ and $l_\diamond(P_\alpha^u \sigma) = \langle 0 = 0 \rangle$. Local correctness is given because $0 = 0$ is a true Δ_0 -formula. Now consider the case $\sigma \in P_\alpha^u$. Then we have $r_\diamond(P_\alpha^u \sigma) = r_\alpha^u(\sigma)$, where r_α^u is the labelling function of the

preproof P_α^u . As an example, assume $r_\alpha^u(\sigma) = (\exists_x, b, \psi)$. By local correctness of P_α^u we get $\sigma \smallfrown 0 \in P_\alpha^u$. It follows that we have

$$o_\smallfrown(n(P_\alpha^u \sigma, 0)) = o_\smallfrown(P_\alpha^u \sigma \smallfrown 0) = o_\alpha^u(\sigma \smallfrown 0),$$

as well as $o_\smallfrown(P_\alpha^u \sigma) = o_\alpha^u(\sigma)$. Thus

$$o_\smallfrown(n(P_\alpha^u \sigma, 0)) + \omega \leq o_\smallfrown(P_\alpha^u \sigma)$$

follows from $o_\alpha^u(\sigma \smallfrown 0) + \omega \leq o_\alpha^u(\sigma)$, as guaranteed by the local correctness of P_α^u . The other conditions and cases are checked in the same way. \square

Corollary 4.7. *For any basic $\mathbf{L}_{\omega^\alpha}^u$ -code P the tree $[P]$ is an $\mathbf{L}_{\omega^\alpha}^u$ -preproof.*

Proof. To check local correctness at a node $\sigma \in [P]$, we distinguish cases according to the rule $r_{[P]}(\sigma) = r(P, \sigma)$. Assume for example that we have $r_{[P]}(\sigma) = (\text{Cut}, \psi)$. By the definition of $[P]$ we have $\sigma \smallfrown a \in [P]$ if and only if $a \in \{0, 1\}$, so this part of the local correctness condition is satisfied. Concerning the remaining conditions, observe that we have

$$r_\smallfrown(\bar{n}(P, \sigma)) = r(P, \sigma) = (\text{Cut}, \psi).$$

Thus condition (L) for $\bar{n}(P, \sigma)$ gives

$$l_\smallfrown(n(\bar{n}(P, \sigma), 0)) \subseteq l_\smallfrown(\bar{n}(P, \sigma)), \psi$$

We can deduce

$$\begin{aligned} l_{[P]}(\sigma \smallfrown 0) &= l(P, \sigma \smallfrown 0) = l_\smallfrown(\bar{n}(P, \sigma \smallfrown 0)) = \\ &= l_\smallfrown(n(\bar{n}(P, \sigma), 0)) \subseteq l_\smallfrown(\bar{n}(P, \sigma)), \psi = l_{[P]}(\sigma), \psi, \end{aligned}$$

as required for the local correctness of $[P]$ at σ . The other conditions are deduced in the same way. \square

Note that Lemma 4.6 only involves the functions $l_\smallfrown, r_\smallfrown, o_\smallfrown, n$ from Definition 4.3. This gives us an easy way to extend the system of basic $\mathbf{L}_{\omega^\alpha}^u$ -codes by new codes: All we need to do is extend the functions $l_\smallfrown, r_\smallfrown, o_\smallfrown, n$ to the new codes and show that condition (L) is still satisfied. Definition 4.4 will automatically provide an interpretation of the new codes (based on the extended functions $l_\smallfrown, r_\smallfrown, o_\smallfrown, n$). The proof of the corollary ensures that the interpretations of the new codes are locally correct $\mathbf{L}_{\omega^\alpha}^u$ -preproofs. As a first application, let us show how a proof of a universal statement $\forall_x \psi(x)$ can be transformed into a proof of any instance $\psi(a)$, keeping the same ordinal height; similarly, a proof of a conjunction can be transformed into a proof of either conjunct:

Lemma 4.8. *We can extend the system of basic $\mathbf{L}_{\omega^\alpha}^u$ -codes in the following way:*

- (a) *For each universal formula $\forall_x \psi$ and each $b \in \mathbf{L}_{\omega^\alpha}^u$ we can add a unary function symbol $\mathcal{I}_{\forall_x \psi, b}$ such that we have*

$$\begin{aligned} l_\smallfrown(\mathcal{I}_{\forall_x \psi, b} P) &= (l_\smallfrown(P) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\}, \\ o_\smallfrown(\mathcal{I}_{\forall_x \psi, b} P) &= o_\smallfrown(P) \end{aligned}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -code P (of the extended system).

(b) For each conjunction $\psi_0 \wedge \psi_1$ and each $i \in \{0, 1\}$ we can add a unary function symbol $\mathcal{I}_{\psi_0 \wedge \psi_1, i}$ such that we have

$$\begin{aligned} l_{\langle \rangle}(\mathcal{I}_{\psi_0 \wedge \psi_1, i}P) &= (l_{\langle \rangle}(P) \setminus \{\psi_0 \wedge \psi_1\}) \cup \{\psi_i\}, \\ o_{\langle \rangle}(\mathcal{I}_{\psi_0 \wedge \psi_1, i}P) &= o_{\langle \rangle}(P) \end{aligned}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -code P (of the extended system).

Proof. (a) Informally, the idea is to replace any rule (\forall_x, ψ) , which infers the formula $\forall_x \psi$ from the premises $\psi(a)$ for $a \in \mathbf{L}_{\omega^\alpha}^u$, by the rule (Rep, b), which repeats the premise $\psi(b)$. It is instructive to phrase this as a recursion on P : First, apply the operator $\mathcal{I}_{\forall_x \psi, b}$ to the immediate subproofs $n(P, a)$ of P , to replace any occurrences of the rule (\forall_x, ψ) in these subproofs. Additionally, if (\forall_x, ψ) is the last rule of P then replace it by the rule (Rep, b). Astonishingly, $\mathbf{L}_{\omega^\alpha}^u$ -codes allow us to make this idea formal, even when the preproof P is not well-founded and recursion on P is not available: Formally, we define $l_{\langle \rangle}(P), r_{\langle \rangle}(P), o_{\langle \rangle}(P)$ and $n(P, a)$ by recursion on the length of the term P . Definition 4.3 accounts for a basic code P . Any other code is of the form $\mathcal{I}_{\forall_x \psi, b}P$, for some formula ψ and some parameter b . Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P) &:= \begin{cases} (\text{Rep}, b) & \text{if } r_{\langle \rangle}(P) = (\forall_x, \psi), \\ r_{\langle \rangle}(P) & \text{otherwise,} \end{cases} \\ n(\mathcal{I}_{\forall_x \psi, b}P, a) &:= \mathcal{I}_{\forall_x \psi, b}n(P, a). \end{aligned}$$

The recursive clauses for $l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)$ and $o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)$ can be copied from the statement of the lemma. We have thus extended $l_{\langle \rangle}(\cdot), r_{\langle \rangle}(\cdot), o_{\langle \rangle}(\cdot)$ and $n(\cdot, \cdot)$ to primitive recursive functions on all $\mathbf{L}_{\omega^\alpha}^u$ -codes of the extended system. It remains to show that condition (L) holds for all new $\mathbf{L}_{\omega^\alpha}^u$ -codes. Note that the statement “condition (L) holds for a given code P ” is primitive recursive (even the quantification over all $a \in \mathbf{L}_{\omega^\alpha}^u$ in the case $r_{\langle \rangle}(P) = (\forall_y, \varphi)$ is harmless — in contrast to the situation for first order number theory, where one has to be careful to avoid an unbounded quantification over ω at this place). We may thus establish condition (L) by induction on the length of $\mathbf{L}_{\omega^\alpha}^u$ -codes. For basic codes condition (L) holds by Lemma 4.6. In the induction step we must deduce condition (L) for the term $\mathcal{I}_{\forall_x \psi, b}P$ from condition (L) for P . We distinguish cases according to the rule $r_{\langle \rangle}(P)$:

Case $r_{\langle \rangle}(P) = \text{Ax}$: Then we have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P) = \text{Ax}$, so we must ensure that

$$l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P) = (l_{\langle \rangle}(P) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\}$$

contains a true Δ_0 -formula. By induction hypothesis $l_{\langle \rangle}(P)$ contains a true Δ_0 -formula. This formula is still contained in $l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)$, unless it is the formula $\forall_x \psi$ itself. In the latter case $\psi(b)$ is also a true Δ_0 -formula, contained in $l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)$.

Case $r_{\langle \rangle}(P) = (\wedge, \psi_0, \psi_1)$: Then we have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P) = (\wedge, \psi_0, \psi_1)$. Using the induction hypothesis we get

$$o_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b}P, i)) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}n(P, i)) = o_{\langle \rangle}(n(P, i)) < o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P).$$

The local correctness of P also gives $\psi_0 \wedge \psi_1 \in l_{\langle \rangle}(P)$. As $\psi_0 \wedge \psi_1$ and $\forall_x \psi$ are different formulas we still have $\psi_0 \wedge \psi_1 \in l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)$. Finally, we also have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b}P, i)) &= l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}n(P, i)) = \\ &= (l_{\langle \rangle}(n(P, i)) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} \subseteq ((l_{\langle \rangle}(P) \cup \{\psi_i\}) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P) \setminus \{\forall_x \psi\}) \cup \{\psi(b), \psi_i\} = l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P) \cup \{\psi_i\}, \end{aligned}$$

as demanded by condition (L) for $\mathcal{I}_{\forall_x \psi, b} P$.

Case $r_{\langle \rangle}(P) = (\forall_i, \psi_0, \psi_1)$: Similar to the previous case.

Case $r_{\langle \rangle}(P) = (\forall_y, \varphi) \neq (\forall_x, \psi)$: Then we have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) = (\forall_y, \varphi)$. By the induction hypothesis we get

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b} P, a)) + \omega &= o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} n(P, a)) + \omega = \\ &= o_{\langle \rangle}(n(P, a)) + \omega \leq o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) \end{aligned}$$

for all $a \in \mathbf{L}_{\omega^\alpha}^u$. Also, the formula $\forall_y \varphi$ occurs in $l_{\langle \rangle}(P)$; as this formula is different from the formula $\forall_x \psi$ it still occurs in $l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P)$. Finally we have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b} P, a)) &= l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} n(P, a)) = \\ &= (l_{\langle \rangle}(n(P, a)) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} \subseteq ((l_{\langle \rangle}(P) \cup \{\varphi(a)\}) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P) \setminus \{\forall_x \psi\}) \cup \{\psi(b), \varphi(a)\} = l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) \cup \{\varphi(a)\}, \end{aligned}$$

as required for condition (L).

Case $r_{\langle \rangle}(P) = (\forall_x, \psi)$: Here we have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) = (\text{Rep}, b)$. We deduce

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b} P, b)) &= o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} n(P, b)) = \\ &= o_{\langle \rangle}(n(P, b)) < o_{\langle \rangle}(n(P, b)) + \omega \leq o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) \end{aligned}$$

from the induction hypothesis. Furthermore we have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{I}_{\forall_x \psi, b} P, b)) &= l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} n(P, b)) = \\ &= (l_{\langle \rangle}(n(P, b)) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} \subseteq ((l_{\langle \rangle}(P) \cup \{\psi(b)\}) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} = \\ &= (l_{\langle \rangle}(P) \setminus \{\forall_x \psi\}) \cup \{\psi(b)\} = l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P), \end{aligned}$$

as condition (L) requires in case of the rule (Rep, b).

Case $r_{\langle \rangle}(P) = (\exists_x, c, \varphi)$: Then we have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) = (\exists_x, c, \varphi)$. From the induction hypothesis we get $|c| < o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P)$. The remaining conditions are checked as in the previous cases.

Case $r_{\langle \rangle}(P) = (\text{Cut}, \varphi)$: Similar to the previous cases.

Case $r_{\langle \rangle}(P) = (\text{Ref}, \exists_z \forall_{v \in a} \exists_{w \in z} \theta)$: We have $r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P) = (\text{Ref}, \exists_z \forall_{v \in a} \exists_{w \in z} \theta)$. Using the induction hypothesis we get $\Omega \leq o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P)$. Also by the induction hypothesis we know that $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ occurs in $l_{\langle \rangle}(P)$. Crucially, the formulas $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ and $\forall_x \psi$ cannot be the same, so that $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ still occurs in $l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b} P)$. The remaining conditions are checked as in the previous cases.

Case $r_{\langle \rangle}(P) = (\text{Rep}, c)$: Similar to the previous cases.

We have established condition (L) for all $\mathbf{L}_{\omega^\alpha}^u$ -codes of the extended system. By (the proof of) Corollary 4.7 it follows that the interpretations of these codes are locally correct $\mathbf{L}_{\omega^\alpha}^u$ -preproofs.

(b) Similar to (a) we set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{I}_{\psi_0 \wedge \psi_1, i} P) &:= \begin{cases} (\text{Rep}, i) & \text{if } r_{\langle \rangle}(P) = (\wedge, \psi_0, \psi_1), \\ r_{\langle \rangle}(P) & \text{otherwise,} \end{cases} \\ n(\mathcal{I}_{\psi_0 \wedge \psi_1, i} P, a) &:= \mathcal{I}_{\psi_0 \wedge \psi_1, i} n(P, a). \end{aligned}$$

The verification of condition (L) is similar to part (a), and left to the reader. Note that we also want to allow codes of the form $\mathcal{I}_{\forall_x \psi, b} \mathcal{I}_{\psi_0 \wedge \psi_1, i} P$. Thus we must repeat the recursive clauses from (a) after adding the symbols $\mathcal{I}_{\psi_0 \wedge \psi_1, i}$. Similarly, we must repeat the arguments from (a) in the inductive verification of condition (L). From a

formal viewpoint it might in fact be preferable to write down the recursive clauses and inductive verifications for all $\mathbf{L}_{\omega^\alpha}^u$ -codes that we ever want to introduce at a single blow. However, a modular presentation seems to be more readable, and can clearly be converted into a formal proof in principle. \square

We have developed a system of codes P that allow us to construct various $\mathbf{L}_{\omega^\alpha}^u$ -preproofs $[P]$ in a primitive recursive way. Now let us investigate how we can prove properties of these preproofs, at the example of cut-rank.

Definition 4.9. We define an assignment of cut ranks

$$d : \text{"}\mathbf{L}_{\omega^\alpha}^u\text{-codes"} \rightarrow \omega$$

by recursion over $\mathbf{L}_{\omega^\alpha}^u$ -codes: For basic codes $P_\alpha^u \sigma$ we set $d(P_\alpha^u \sigma) = C$, where C is the constant from Lemma 4.2. We extend d to the $\mathbf{L}_{\omega^\alpha}^u$ -codes introduced in Lemma 4.8 by setting

$$d(\mathcal{I}_{\forall x \psi, b} P) := d(\mathcal{I}_{\psi_0 \wedge \psi_1, i} P) := d(P).$$

So far, the assignment of cut ranks to $\mathbf{L}_{\omega^\alpha}^u$ -codes is constant; more interesting cases will appear shortly. The following is parallel to the development in Lemma 4.6 and Corollary 4.7:

Lemma 4.10. *The assignment of cut ranks to $\mathbf{L}_{\omega^\alpha}^u$ -codes is locally correct, in the sense that the following conditions hold for any $\mathbf{L}_{\omega^\alpha}^u$ -code P :*

- (C1) *If $r_\diamond(P)$ is of the form (Cut, φ) then we have $\text{ht}(\varphi) < d(P)$.*
- (C2) *We have $d(n(P, a)) \leq d(P)$ for all $a \in \iota(r_\diamond(P))$.*

Proof. We argue by induction on (the length of) $\mathbf{L}_{\omega^\alpha}^u$ -codes. First consider the case of a basic code $P = P_\alpha^u \sigma$. Concerning (C1) we need to distinguish two cases: If σ is a node in P_α^u then we have $r_\diamond(P_\alpha^u \sigma) = r_\alpha^u(\sigma)$, where r_α^u is the labelling function of the preproof P_α^u . In this case (C1) holds by Lemma 4.2. If σ lies outside of P_α^u then we have $r_\diamond(P_\alpha^u \sigma) = \text{Ax}$ by definition, so (C1) does not apply. Concerning (C2) for the basic code $P = P_\alpha^u \sigma$, observe that we have

$$d(n(P_\alpha^u \sigma, a)) = d(P_\alpha^u \sigma \frown a) = C$$

because $P_\alpha^u \sigma \frown a$ is also a basic code. In the induction step, consider $P = \mathcal{I}_{\forall x \psi, b} P'$. Note that $r_\diamond(\mathcal{I}_{\forall x \psi, b} P') = (\text{Cut}, \varphi)$ can only occur if we have $r_\diamond(P') = (\text{Cut}, \varphi)$. Using the induction hypothesis we thus get

$$\text{ht}(\varphi) < d(P') = d(\mathcal{I}_{\forall x \psi, b} P'),$$

as needed for (C1). Concerning (C2), observe that $\iota(r_\diamond(\mathcal{I}_{\forall x \psi, b} P')) \subseteq \iota(r_\diamond(P'))$ holds in all possible cases. Thus the induction hypothesis implies

$$d(n(\mathcal{I}_{\forall x \psi, b} P', a)) = d(\mathcal{I}_{\forall x \psi, b} n(P', a)) = d(n(P', a)) \leq d(P') = d(\mathcal{I}_{\forall x \psi, b} P')$$

for all $a \in \iota(r_\diamond(\mathcal{I}_{\forall x \psi, b} P'))$. For a code $P = \mathcal{I}_{\psi_0 \wedge \psi_1, u} P'$ one argues similarly. \square

Corollary 4.11. *For any $\mathbf{L}_{\omega^\alpha}^u$ -code P the $\mathbf{L}_{\omega^\alpha}^u$ -preproof $[P]$ has cut-rank $d(P)$.*

Proof. Recall the function \bar{n} defined by the recursion

$$\begin{aligned} \bar{n}(P, \langle \rangle) &= P, \\ \bar{n}(P, \sigma \frown a) &= n(\bar{n}(P, \sigma), a). \end{aligned}$$

By induction on the sequence $\sigma \in [P]$ one establishes $d(\bar{n}(P, \sigma)) \leq d(P)$, using condition (C2) from the previous lemma as the induction step. Also recall that

$r_{[P]}(\sigma)$ was defined to be the rule $r_{\langle \rangle}(\bar{n}(P, \sigma))$. Thus if $r_{[P]}(\sigma)$ is of the form (Cut, φ) then we have

$$\text{ht}(\varphi) < d(\bar{n}(P, \sigma)) \leq d(P),$$

by condition (C1) from the previous lemma. \square

It turns out that the corollary that we have just established is never used. This is because cut-rank is an auxiliary notion, which we could limit to the realm of codes rather than actual proofs. To handle cut-ranks for codes we will need Lemma 4.10 but not, strictly speaking, its corollary. In contrast to this, Corollary 4.7 above will be used, namely to obtain an actual $\mathbf{L}_{\omega^\alpha}^u$ -preproof to which Proposition 3.4 can be applied (of course we could renounce the notion of $\mathbf{L}_{\omega^\alpha}^u$ -preproof altogether, and formulate everything in terms of codes — but this seems somewhat forced in a set-theoretic context, where infinite proof trees exist as first-rate objects). Even though they are not strictly required we will continue to state results such as Corollary 4.11, as they clarify the intended semantics of $\mathbf{L}_{\omega^\alpha}^u$ -codes.

In the following we will extend our system of $\mathbf{L}_{\omega^\alpha}^u$ -codes by a unary function symbol \mathcal{E} such that the $\mathbf{L}_{\omega^\alpha}^u$ -preproof $[\mathcal{E}P]$ has lower cut-rank than $[P]$. To show that this is the case we will only need to extend the assignment d from Definition 4.9 to the new codes. We will prove that this extension of d satisfies the conditions (C1) and (C2). By the (proof of the) corollary it will immediately follow that the preproof $[\mathcal{E}P]$ has cut-rank $< d(\mathcal{E}P)$. The following is a preparation:

Lemma 4.12. *We can extend the system of $\mathbf{L}_{\omega^\alpha}^u$ -codes in the following way:*

- (a) *For each formula $\exists_x \psi$ with $\text{ht}(\exists_x \psi) > 1$ we add a binary function symbol $\mathcal{R}_{\exists_x \psi}$ such that we have*

$$\begin{aligned} l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) &= (l_{\langle \rangle}(P_0) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}), \\ o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) &= o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0), \\ d(\mathcal{R}_{\exists_x \psi} P_0 P_1) &= \max\{d(P_0), d(P_1), \text{ht}(\exists_x \psi)\} \end{aligned}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -codes P_0, P_1 .

- (b) *For each formula $\psi_0 \vee \psi_1$ with $\text{ht}(\psi_0 \vee \psi_1) > 1$ we add a binary function symbol $\mathcal{R}_{\psi_0 \vee \psi_1}$ such that we have*

$$\begin{aligned} l_{\langle \rangle}(\mathcal{R}_{\psi_0 \vee \psi_1} P_0 P_1) &= (l_{\langle \rangle}(P_0) \setminus \{\psi_0 \vee \psi_1\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\neg \psi_0 \wedge \neg \psi_1\}), \\ o_{\langle \rangle}(\mathcal{R}_{\psi_0 \vee \psi_1} P_0 P_1) &= o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0), \\ d(\mathcal{R}_{\psi_0 \vee \psi_1} P_0 P_1) &= \max\{d(P_0), d(P_1), \text{ht}(\psi_0 \vee \psi_1)\} \end{aligned}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -codes P_0, P_1 .

Proof. (a) Let us first describe the proof idea in informal terms: Assume that the formula $\exists_x \psi$ is deduced from a premise $\psi(b)$ at some node of P . We want to avoid the introduction of $\exists_x \psi$, to keep it out of the end-sequent of $\mathcal{R}_{\exists_x \psi} P_0 P_1$. To achieve this we transform P_1 according to Lemma 4.8, deleting the formula $\forall_x \neg \psi$ in favour of $\neg \psi(b)$. Now we can apply a cut over $\psi(b)$, to remove the premise $\psi(b)$ in P_0 and the formula $\neg \psi(b)$ that we have added to P_1 . Concerning the cut-rank, note that we have $\text{ht}(\psi(b)) < \text{ht}(\exists_x \psi)$. It is helpful to phrase this as a recursion over the preproof P_0 : First, form the preproofs $\mathcal{R}_{\exists_x \psi} n(P_0, a) P_1$, to remove the formula $\exists_x \psi$ from the immediate subtrees $n(P_0, a)$ of P_0 . If $\exists_x \psi$ was introduced by the last

rule of P_0 , then remove it by a cut over $\psi(b)$, as described above. Formally, we extend the functions $l_{\langle \rangle}, r_{\langle \rangle}, o_{\langle \rangle}, n, d$ to the new codes by the recursive clauses

$$\begin{aligned} r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) &:= \begin{cases} (\text{Cut}, \psi(b)) & \text{if } r_{\langle \rangle}(P_0) = (\exists x, b, \psi) \text{ for some } b, \\ r_{\langle \rangle}(P_0) & \text{otherwise,} \end{cases} \\ n(\mathcal{R}_{\exists x \psi} P_0 P_1, a) &:= \begin{cases} \mathcal{I}_{\forall x \neg \psi, b} P_1 & \text{if } r_{\langle \rangle}(P_0) = (\exists x, b, \psi) \text{ and } a = 1, \\ \mathcal{R}_{\exists x \psi} n(P_0, a) P_1 & \text{otherwise.} \end{cases} \end{aligned}$$

Conditions (L), (C1) and (C2) are verified by induction on the length of the new $\mathbf{L}_{\omega^\alpha}^u$ -codes: Concerning (C1), assume that $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1)$ is of the form (Cut, φ) . There are two possibilities: If φ is a formula $\psi(b)$ then we have

$$\text{ht}(\varphi) < \text{ht}(\exists x \psi) \leq d(\mathcal{R}_{\exists x \psi} P_0 P_1),$$

as required. Otherwise we must have $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) = r_{\langle \rangle}(P_0)$. Using the induction hypothesis for P_0 we get

$$\text{ht}(\varphi) < d(P_0) \leq d(\mathcal{R}_{\exists x \psi} P_0 P_1).$$

As for condition (C2), we either have

$$d(n(\mathcal{R}_{\exists x \psi} P_0 P_1, a)) = d(\mathcal{I}_{\forall x \neg \psi, b} P_1) = d(P_1) \leq d(\mathcal{R}_{\exists x \psi} P_0 P_1)$$

or, using the induction hypothesis for P_0 ,

$$\begin{aligned} d(n(\mathcal{R}_{\exists x \psi} P_0 P_1, a)) &= d(\mathcal{R}_{\exists x \psi} s(P_0, a) P_1) = \max\{d(s(P_0, a)), d(P_1), \text{ht}(\exists x \psi)\} \leq \\ &\leq \max\{d(P_0), d(P_1), \text{ht}(\exists x \psi)\} = d(\mathcal{R}_{\exists x \psi} P_0 P_1). \end{aligned}$$

To verify condition (L) we distinguish cases according to the rule $r_{\langle \rangle}(P_0)$:

Case $r_{\langle \rangle}(P_0) = \text{Ax}$: Then we have $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) = \text{Ax}$. Condition (L) for P_0 tells us that $l_{\langle \rangle}(P_0)$ contains a true Δ_0 -formula. Our assumption $\text{ht}(\exists x \psi) > 1$ implies that $\exists x \psi$ is not a Δ_0 -formula. Thus the true Δ_0 -formula is still contained in $l_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1)$, as required by condition (L) for $\mathcal{R}_{\exists x \psi} P_0 P_1$.

Case $r_{\langle \rangle}(P_0) = (\wedge, \psi_0, \psi_1)$: Then we have $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) = (\wedge, \psi_0, \psi_1)$. By condition (L) for P_0 we get

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{R}_{\exists x \psi} P_0 P_1, i)) &= o_{\langle \rangle}(\mathcal{R}_{\exists x \psi} s(P_0, i) P_1) = \\ &= o_{\langle \rangle}(P_1) + o_{\langle \rangle}(s(P_0, i)) < o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0) = o_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) \end{aligned}$$

for $i = 0, 1$. Condition (L) also tells us that $\psi_0 \wedge \psi_1$ occurs in $l_{\langle \rangle}(P_0)$. As the formula $\psi_0 \wedge \psi_1$ is different from $\exists x \psi$ it still occurs in $l_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1)$. Finally, again using condition (L) for P_0 , we have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{R}_{\exists x \psi} P_0 P_1, i)) &= l_{\langle \rangle}(\mathcal{R}_{\exists x \psi} s(P_0, i) P_1) = \\ &= (l_{\langle \rangle}(s(P_0, i)) \setminus \{\exists x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall x \neg \psi\}) \subseteq \\ &\subseteq ((l_{\langle \rangle}(P_0) \cup \{\psi_i\}) \setminus \{\exists x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall x \neg \psi\}) \subseteq \\ &\subseteq (l_{\langle \rangle}(P_0) \setminus \{\exists x \psi\}) \cup \{\psi_i\} \cup (l_{\langle \rangle}(P_1) \setminus \{\forall x \neg \psi\}) = l_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) \cup \{\psi_i\}, \end{aligned}$$

as required by condition (L) for $\mathcal{R}_{\exists x \psi} P_0 P_1$.

Case $r_{\langle \rangle}(P_0) = (\vee, \psi_0, \psi_1)$: Similar to the previous case.

Case $r_{\langle \rangle}(P_0) = (\forall y, \varphi)$: Similar to the previous cases.

Case $r_{\langle \rangle}(P_0) = (\exists y, b, \varphi)$ with $\exists y \varphi \neq \exists x \psi$: Then $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi} P_0 P_1) = (\exists y, b, \varphi)$. By condition (L) for P_0 the formula $\exists y \varphi$ occurs in $l_{\langle \rangle}(P_0)$. In view of $\exists y \varphi \neq \exists x \psi$ this

formula is still contained in $l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1)$. The remaining conditions are checked as in the previous cases.

Case $r_{\langle \rangle}(P_0) = (\exists_x, b, \psi)$: Then we have $r_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) = (\text{Cut}, \psi(b))$. We have

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, 0)) &= o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} s(P_0, 0) P_1) = \\ &= o_{\langle \rangle}(P_1) + o_{\langle \rangle}(s(P_0, 0)) < o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0) = o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) \end{aligned}$$

and

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, 1)) &= o_{\langle \rangle}(\mathcal{I}_{\forall_x \neg \psi, b} P_1) = o_{\langle \rangle}(P_1) \leq \\ &\leq o_{\langle \rangle}(P_1) + o_{\langle \rangle}(s(P_0, 0)) < o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0) = o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1). \end{aligned}$$

Condition (L) for P_0 yields

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, 0)) &= l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} n(P_0, 0) P_1) = \\ &= (l_{\langle \rangle}(n(P_0, 0)) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}) \subseteq \\ &\subseteq ((l_{\langle \rangle}(P_0) \cup \{\psi(b)\}) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}) \subseteq \\ &\subseteq (l_{\langle \rangle}(P_0) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}) \cup \{\psi(b)\} = l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) \cup \{\psi(b)\}. \end{aligned}$$

By Lemma 4.8 we have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, 1)) &= l_{\langle \rangle}(\mathcal{I}_{\forall_x \neg \psi, b} P_1) = (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}) \cup \{\neg \psi(b)\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P_0) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(P_1) \setminus \{\forall_x \neg \psi\}) \cup \{\neg \psi(b)\} = l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) \cup \{\neg \psi(b)\}, \end{aligned}$$

as condition (L) requires in the case of the rule $(\text{Cut}, \psi(b))$.

Case $r_{\langle \rangle}(P_0) = (\text{Ref}, \exists_z \forall_{v \in a} \exists_{w \in z} \theta)$: Then $r_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1)$ is the same rule. We have

$$\Omega \leq o_{\langle \rangle}(P_0) \leq o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0) = o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1).$$

Condition (L) for P_0 implies that $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ occurs in $l_{\langle \rangle}(P_0)$. Since θ is a Δ_0 -formula we have $\text{ht}(\exists_z \forall_{v \in a} \exists_{w \in z} \theta) = 1$. Thus our assumption $\text{ht}(\exists_x \psi) > 1$ implies that $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ and $\exists_x \psi$ are different formulas. It follows that $\exists_z \forall_{v \in a} \exists_{w \in z} \theta$ is still contained in $l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1)$, as demanded by condition (L) in the case of the rule $(\text{Ref}, \exists_z \forall_{v \in a} \exists_{w \in z} \theta)$. The other conditions are verified as in the previous cases.

Case $r_{\langle \rangle}(P_0) = (\text{Rep}, b)$: Similar to the previous cases.

(b) Similar to part (a) we set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{R}_{\psi_0 \vee \psi_1} P_0 P_1) &:= \begin{cases} (\text{Cut}, \psi_i) & \text{if } r_{\langle \rangle}(P_0) = (\vee_i, \psi_0, \psi_1) \text{ for some } i \in \{0, 1\}, \\ r_{\langle \rangle}(P_0) & \text{otherwise,} \end{cases} \\ n(\mathcal{R}_{\psi_0 \vee \psi_1} P_0 P_1, a) &:= \begin{cases} \mathcal{I}_{\neg \psi_0 \wedge \neg \psi_1, i} P_1 & \text{if } r_{\langle \rangle}(P_0) = (\vee_i, \psi_0, \psi_1) \text{ and } a = 1, \\ \mathcal{R}_{\psi_0 \vee \psi_1} s(P_0, a) P_1 & \text{otherwise.} \end{cases} \end{aligned}$$

The verification of (L), (C1) and (C2) is similar to (a), and left to the reader. \square

Finally we have all ingredients for the desired cut elimination operator:

Proposition 4.13. *We can extend the system of $\mathbf{L}_{\omega\alpha}^u$ -codes by a unary function symbol \mathcal{E} such that we have*

$$\begin{aligned} l_{\langle \rangle}(\mathcal{E}P) &= l_{\langle \rangle}(P), \\ o_{\langle \rangle}(\mathcal{E}P) &= \Omega^{o_{\langle \rangle}(P)}, \\ d(\mathcal{E}P) &= \max\{2, d(P) - 1\} \end{aligned}$$

for each $\mathbf{L}_{\omega^\alpha}^u$ -code P .

Proof. The intuitive idea is straightforward: We replace any cut over a formula $\exists_x \psi$ by an application of the operator $\mathcal{R}_{\exists_x \psi}$, which only involves cuts of lower rank. Formally, the functions $l_\Diamond(P)$, $r_\Diamond(P)$, $o_\Diamond(P)$, $n(P, a)$, $d(P)$ are defined by recursion on the length of the code P . We distinguish cases according to the last rule of P , and verify conditions (L), (C1) and (C2) as we go along:

Case $r_\Diamond(P) = \text{Ax}$: We set $r_\Diamond(\mathcal{E}P) := \text{Ax}$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$ (the latter is irrelevant because of $\iota(\text{Ax}) = \emptyset$). By the induction hypothesis $l_\Diamond(\mathcal{E}P) = l_\Diamond(P)$ contains a true Δ_0 -formula, as required for condition (L). Conditions (C1) and (C2) do not apply.

Case $r_\Diamond(P) = (\wedge, \psi_0, \psi_1)$: Set $r_\Diamond(\mathcal{E}P) := (\wedge, \psi_0, \psi_1)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. Let us verify condition (L): As exponentiation to the base Ω is monotone we get

$$o(n(\mathcal{E}P, i)) = o(\mathcal{E}n(P, i)) = \Omega^{o(n(P, i))} < \Omega^{o(P)} = o(\mathcal{E}P).$$

By condition (L) for P the formula $\psi_0 \wedge \psi_1$ is contained in $l_\Diamond(P) = l_\Diamond(\mathcal{E}P)$. Also, we have

$$l_\Diamond(n(\mathcal{E}P, i)) = l_\Diamond(\mathcal{E}n(P, i)) = l_\Diamond(n(P, i)) \subseteq l_\Diamond(P) \cup \{\psi_i\} = l_\Diamond(\mathcal{E}P) \cup \{\psi_i\}$$

for $i = 0, 1$, as required by condition (L) for $\mathcal{E}P$. Condition (C1) does not apply. Using (C2) for P we get

$$d(n(\mathcal{E}P, i)) = d(\mathcal{E}n(P, i)) = \max\{2, d(n(P, i)) - 1\} \leq \max\{2, d(P) - 1\} = d(\mathcal{E}P),$$

as required by condition (C2) for $\mathcal{E}P$.

Case $r_\Diamond(P) = (\vee_i, \psi_0, \psi_1)$: Set $r_\Diamond(\mathcal{E}P) := (\vee_i, \psi_0, \psi_1)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. The verification of (L), (C1) and (C2) is similar to the previous case.

Case $r_\Diamond(P) = (\forall_x, \psi)$: Set $r_\Diamond(\mathcal{E}P) := (\forall_x, \psi)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. Concerning the ordinal height, we have

$$o(n(\mathcal{E}P, a)) + \omega = \Omega^{o(n(P, a))} + \omega < \Omega^{o(n(P, a)) + 1} \leq \Omega^{o(P)} = o(\mathcal{E}P).$$

The other conditions are checked as in the previous cases.

Case $r_\Diamond(P) = (\exists_x, b, \psi)$: Set $r_\Diamond(\mathcal{E}P) := (\exists_x, b, \psi)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. Observe that we have

$$|b| < o(P) \leq \Omega^{o(P)} = o(\mathcal{E}P),$$

as demanded by condition (L). The other conditions are verified as in above.

Case $r_\Diamond(P) = (\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta)$: We set $r_\Diamond(\mathcal{E}P) := (\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. Observe that we have

$$\Omega \leq o(P) \leq \Omega^{o(P)} = o(\mathcal{E}P).$$

The other conditions are checked as in the previous cases.

Case $r_\Diamond(P) = (\text{Rep}, b)$: Set $r_\Diamond(\mathcal{E}P) := (\text{Rep}, b)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. The verification of (L), (C1) and (C2) is similar to the previous cases.

Case $r_\Diamond(P) = (\text{Cut}, \psi)$ with $\text{ht}(\psi) \leq 1$: In this case we set $r_\Diamond(\mathcal{E}P) := (\text{Cut}, \psi)$ and $n(\mathcal{E}P, a) := \mathcal{E}n(P, a)$. Condition (C1) follows from $\text{ht}(\psi) \leq 1 < 2 \leq d(\mathcal{E}P)$. Conditions (L) and (C2) are checked as in the previous cases.

Case $r_\Diamond(P) = (\text{Cut}, \exists_x \psi)$ with $\text{ht}(\exists_x \psi) > 1$: Here we set

$$r_\Diamond(\mathcal{E}P) := (\text{Rep}, 0),$$

$$n(\mathcal{E}P, a) := \mathcal{R}_{\exists_x \psi}(\mathcal{E}n(P, 0))(\mathcal{E}n(P, 1)).$$

Let us verify condition (L): Using Lemma 4.12 and condition (L) for P we get

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{E}P, 0)) &= o_{\langle \rangle}(\mathcal{E}n(P, 1)) + o_{\langle \rangle}(\mathcal{E}n(P, 0)) = \\ &= \Omega^{o_{\langle \rangle}(n(P, 1))} + \Omega^{o_{\langle \rangle}(n(P, 0))} < \Omega^{o_{\langle \rangle}(P)} = o_{\langle \rangle}(\mathcal{E}P). \end{aligned}$$

Also, we have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{E}P, 0)) &= l_{\langle \rangle}(\mathcal{R}_{\exists_x \psi}(\mathcal{E}n(P, 0))(\mathcal{E}n(P, 1))) = \\ &= (l_{\langle \rangle}(\mathcal{E}n(P, 0)) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(\mathcal{E}n(P, 1)) \setminus \{\forall_x \neg \psi\}) = \\ &= (l_{\langle \rangle}(n(P, 0)) \setminus \{\exists_x \psi\}) \cup (l_{\langle \rangle}(n(P, 1)) \setminus \{\forall_x \neg \psi\}) \subseteq l_{\langle \rangle}(P) = l_{\langle \rangle}(\mathcal{E}P). \end{aligned}$$

Here, the last line relies on condition (L) for P , which provides the inclusions $l_{\langle \rangle}(n(P, 0)) \subseteq l_{\langle \rangle}(P) \cup \{\exists_x \psi\}$ and $l_{\langle \rangle}(n(P, 1)) \subseteq l_{\langle \rangle}(P) \cup \{\forall_x \neg \psi\}$. Condition (C1) does not apply. Condition (C2) holds by

$$\begin{aligned} d(n(\mathcal{E}P, 0)) &= \max\{d(\mathcal{E}n(P, 0)), d(\mathcal{E}n(P, 1)), \text{ht}(\exists_x \psi)\} = \\ &= \max\{2, d(n(P, 0)) - 1, d(n(P, 1)) - 1, \text{ht}(\exists_x \psi)\} = \\ &= \max\{2, d(P) - 1, \text{ht}(\exists_x \psi)\} = \max\{2, d(P) - 1\} = d(\mathcal{E}P). \end{aligned}$$

The last line uses $\text{ht}(\exists_x \psi) < d(P)$, as provided by condition (C1) for P .

Case $r_{\langle \rangle}(P) = (\text{Cut}, \forall_x \psi)$ with $\text{ht}(\forall_x \psi) > 1$: We set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{E}P) &:= (\text{Rep}, 0), \\ n(\mathcal{E}P, a) &:= \mathcal{R}_{\exists_x \neg \psi}(\mathcal{E}n(P, 1))(\mathcal{E}n(P, 0)). \end{aligned}$$

The verification of conditions (L), (C1) and (C2) is similar to the previous case.

Case $r_{\langle \rangle}(P) = (\text{Cut}, \psi_0 \vee \psi_1)$ with $\text{ht}(\psi_0 \vee \psi_1) > 1$: Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{E}P) &:= (\text{Rep}, 0), \\ n(\mathcal{E}P, a) &:= \mathcal{R}_{\psi_0 \vee \psi_1}(\mathcal{E}n(P, 0))(\mathcal{E}n(P, 1)). \end{aligned}$$

The verification of conditions (L), (C1) and (C2) is similar to the previous cases.

Case $r_{\langle \rangle}(P) = (\text{Cut}, \psi_0 \wedge \psi_1)$ with $\text{ht}(\psi_0 \wedge \psi_1) > 1$: Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{E}P) &:= (\text{Rep}, 0), \\ n(\mathcal{E}P, a) &:= \mathcal{R}_{\neg \psi_0 \vee \neg \psi_1}(\mathcal{E}n(P, 1))(\mathcal{E}n(P, 0)). \end{aligned}$$

The verification of conditions (L), (C1) and (C2) is similar to the previous cases. \square

In the following we write $\mathcal{E}^C P$ for the $\mathbf{L}_{\omega^\alpha}^u$ -code $\mathcal{E} \cdots \mathcal{E}P$ with C occurrences of the function symbol \mathcal{E} .

Proposition 4.14. *Let $C \in \omega$ be as in Lemma 4.2. The $\mathbf{L}_{\omega^\alpha}^u$ -preproof $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ has empty end-sequent, height $\varepsilon_{\langle \rangle} \in \varepsilon(S_{\omega^\alpha}^u)$, and cut-rank 2.*

Proof. Using Proposition 4.13 and Proposition 3.3 we get

$$l_{[\mathcal{E}^C P_\alpha^u \langle \rangle]}(\langle \rangle) = l_{\langle \rangle}(\mathcal{E}^C P_\alpha^u \langle \rangle) = l_{\langle \rangle}(P_\alpha^u \langle \rangle) = l_\alpha^u(\langle \rangle) = \langle \rangle,$$

which means that $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ has empty end-sequent. In view of $\Omega^{\varepsilon_{\langle \rangle}} = \varepsilon_{\langle \rangle}$ a similar argument shows $o_{[\mathcal{E}^C P_\alpha^u \langle \rangle]}(\langle \rangle) = \varepsilon_{\langle \rangle}$. Proposition 4.13 and Definition 4.9 give

$$d(\mathcal{E}^C P_\alpha^u \langle \rangle) = \max\{2, d(P_\alpha^u \langle \rangle) - C\} = 2.$$

By Corollary 4.11 this implies that $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ has cut-rank 2. \square

Let us stress once more that the functions $l_{\langle \rangle}, r_{\langle \rangle}, o_{\langle \rangle}, n$, defined by recursion on the length of $\mathbf{L}_{\omega^\alpha}^u$ -codes, are primitive recursive. Thus a function such as

$$(\sigma, \alpha, u, C) \mapsto l_{[\mathcal{E}^C P_\alpha^u \langle \rangle]}(\sigma) = l_{\langle \rangle}(\bar{n}(\mathcal{E}^C P_\alpha^u \langle \rangle, \sigma))$$

is primitive recursive as well. Let us also point out that no Bachmann-Howard collapse $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$ was needed for the continuous cut-elimination procedure of the present section. This will be different in the next section, where we collapse $\mathbf{L}_{\omega^\alpha}^u$ -preproofs to height below Ω .

5. COLLAPSING PROOFS

Let us recall our overall goal: We want to establish that any countable transitive set u is contained in some admissible set. In the last paragraph of Section 3 we have described the following plan to achieve this: By Proposition 3.3 we have $\mathbf{L}_{\omega^\alpha}^u$ -preproofs P_α^u with empty end-sequent and ordinal height $\varepsilon_{\langle \rangle} > \Omega$. On the other hand, Proposition 3.4 tells us that no $\mathbf{L}_{\omega^\alpha}^u$ -preproof with empty end-sequent can have ordinal height below Ω . Now assume, aiming at a contradiction, that there is no admissible set \mathbb{A} with $u \subseteq \mathbb{A}$. Then Proposition 2.14 implies that $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ is a well-ordering principle. So the higher Bachmann-Howard principle from Definition 2.3 yields a collapsing function $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$, for some ordinal α . Using this function we want to collapse P_α^u to ordinal height below Ω . This is the desired contradiction, showing that u is contained in an admissible set. The previous section was devoted to a preliminary transformation, turning P_α^u into a preproof of cut-rank 2. In this section we will present the collapsing procedure itself.

A particularly elegant description of collapsing relies on the notion of operator control, due to Buchholz [Buc93]. As controlling operators we use the functions

$$\mathcal{H}_t^\vartheta[\beta] : \gamma \mapsto \mathcal{H}_t^\vartheta[\beta, \gamma]$$

introduced in Definition 2.23. Here we assume that $t \in \varepsilon(S_{\omega^\alpha}^u)$ satisfies $\Omega \leq t$, that we have $\beta, \gamma < \omega^\alpha$, and that $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$ is a Bachmann-Howard collapse. In Section 2 we have seen that $\mathcal{H}_t^\vartheta[\beta, \gamma]$ can be computed by a primitive recursive function in $t, \beta, \gamma, \vartheta$ (and u, α). The functions $\mathcal{H}_t^\vartheta[\beta]$, which restrict this primitive recursive class function to a set, will thus exist as sets. Before we can give a definition of operator controlled proof, we must specify which parameters we wish to control:

Definition 5.1. For an $\mathbf{L}_{\omega^\alpha}^u$ -formula ψ we set

$$k(\psi) := \max(\{|a|; \text{the parameter } a \text{ occurs in } \psi\} \cup \{0\}).$$

Given a sequent $\Gamma = \langle \psi_1, \dots, \psi_k \rangle$ we set

$$k(\Gamma) := \max\{k(\psi_1), \dots, k(\psi_k), 0\}.$$

Concerning parameters in rules, we put

$$\begin{aligned} k((\exists_x, a, \psi)) &:= k((\text{Rep}, a)) := |a|, \\ k((\text{Cut}, \psi)) &:= k(\psi), \end{aligned}$$

and $k(r) := 0$ for any rule of a different form. Finally, we put

$$k_P(\sigma) := \max\{o(\sigma)^*, k(l(\sigma)), k(r(\sigma))\}.$$

for an $\mathbf{L}_{\omega^\alpha}^u$ -preproof (P, l, r, o) and a node $\sigma \in P$.

The reader may wish to recall Notation 2.16: In particular $o(\sigma)^*$ refers to the rank function of the well-ordering principle $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$. As we consider $o(\sigma)^*$ rather than $o(\sigma)$ we have $k_P(\sigma) < \omega^\alpha$. We can now say when an operator controls a proof:

Definition 5.2. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. We say that the operator $\mathcal{H}_t^\vartheta[\beta]$ controls the $\mathbf{L}_{\omega^\alpha}^u$ -preproof P if we have

$$k_P(\sigma) \in \mathcal{H}_t^\vartheta[\beta, |\sigma|]$$

for all nodes $\sigma \in P$.

Our first goal is operator control for the “basic” preproofs P_α^u from Proposition 3.3. The following is a preparation (observe $\mathcal{H}_t^\vartheta[0, \gamma] = \mathcal{H}_t^\vartheta[\gamma]$, by Lemma 2.25):

Lemma 5.3. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. The $\mathbf{L}_{\omega^\alpha}^u$ -preproofs of the Kripke-Platek axioms, as constructed in Lemma 3.2, are controlled by the operator $\mathcal{H}_\Omega^\vartheta$.

Proof. As an example, consider a node of the form $\sigma = \langle c_1, \dots, c_k, a \rangle$ in a proof P of Δ_0 -separation. There we have

$$k_P(\sigma) = \max\{\Omega^*, |a|, |b|, |c_1|, \dots, |c_k|\}$$

with $b = \{z \in a \mid \theta(a, z, \vec{c})\}$. We have observed $|b| < \max\{|a|, |c_1|, \dots, |c_k|\} + \omega$ in the proof of Lemma 3.2. Together with $\Omega^* = 0$ we get

$$k_P(\sigma) < \max\{|a|, |c_1|, \dots, |c_k|\} + \omega = |\sigma| + \omega.$$

From Lemma 2.24 we know $|\sigma| \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$. Lemma 2.26 gives $\omega \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$ and then $|\sigma| + \omega \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$. In view of $k_P(\sigma) < |\sigma| + \omega < \Omega$ we get $k_P(\sigma) \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$. The other cases are similar and left to the reader. \square

Building on this, let us look at the preproofs from Proposition 3.3:

Lemma 5.4. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. The $\mathbf{L}_{\omega^\alpha}^u$ -preproof P_α^u is controlled by the operator $\mathcal{H}_\Omega^\vartheta$.

Proof. First, consider a node σ in the search tree $S_{\omega^\alpha}^u \subseteq P_\alpha^u$. Let us begin by looking at the “ordinal” label $o_\alpha^u(\sigma) = \varepsilon_\sigma$. In view of $|\sigma| \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$ Lemma 2.26 gives $\varepsilon_\sigma \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$. By Lemma 2.21 we obtain $\varepsilon_{\sigma^*} \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$. Thus we have established $k_{P_\alpha^u}(\sigma) \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$ in case $k_P(\sigma) = o_\alpha^u(\sigma)^*$. Now let us look at $k(l_\alpha^u(\sigma))$ and $k(r_\alpha^u(\sigma))$: In the proof of Lemma 1.2 we have seen that all parameters in $l_\alpha^u(\sigma)$ lie in the set $\text{rng}(\sigma) \cup u$; the same argument applies to parameters of the rule $r_\alpha^u(\sigma)$. As we have $|u_i| = |u_i|_{\mathbf{L}}^u = 0$ for all $u_i \in u$ we get $\max\{k(l_\alpha^u(\sigma)), k(r_\alpha^u(\sigma))\} \leq |\sigma|$, which implies $\max\{k(l_\alpha^u(\sigma)), k(r_\alpha^u(\sigma))\} \in \mathcal{H}_\Omega^\vartheta[|\sigma|]$.

It remains to look at a node $\sigma \in P_\alpha^u$ which does not lie in the search tree S_α^u . Then σ is of the form $\sigma_0 \frown \sigma_1$ where σ_1 lies in one of the proofs of the Kripke-Platek axioms, constructed in Lemma 3.2. From the previous lemma we get $k_{P_\alpha^u}(\sigma) \in \mathcal{H}_\Omega^\vartheta[|\sigma_1|]$. In view of $|\sigma_1| \leq |\sigma|$ Lemma 2.25 gives $\mathcal{H}_\Omega^\vartheta[|\sigma_1|] \subseteq \mathcal{H}_\Omega^\vartheta[|\sigma|]$, and the claim follows. \square

In the previous section we have developed $\mathbf{L}_{\omega^\alpha}^u$ -codes as an important tool for the construction of $\mathbf{L}_{\omega^\alpha}^u$ -preproofs. We need to recast the notion of operator control in terms of these codes. The bound on parameters is easily defined for codes:

Definition 5.5. For any $\mathbf{L}_{\omega^\alpha}^u$ -code P we set

$$k_{\langle \rangle}(P) := \max\{o_{\langle \rangle}(P)^*, k(l_{\langle \rangle}(P)), k(r_{\langle \rangle}(P))\}.$$

As for the controlling operators, we can of course represent $\mathcal{H}_t^\vartheta[\beta]$ by the pair $\langle t, \beta \rangle$. Let us restrict attention to the basic $\mathbf{L}_{\omega^\alpha}^u$ -codes from Definition 4.3 first:

Definition 5.6. We define an assignment

$$\begin{aligned} h_0 : \text{"basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes"} &\rightarrow \{t \in \varepsilon(S_{\omega^\alpha}^u) \mid \Omega \leq t\}, & h_0(P_\alpha^u \sigma) &:= \Omega, \\ h_1 : \text{"basic } \mathbf{L}_{\omega^\alpha}^u\text{-codes"} &\rightarrow \omega^\alpha, & h_1(P_\alpha^u \sigma) &:= |\sigma| \end{aligned}$$

of operators to basic $\mathbf{L}_{\omega^\alpha}^u$ -codes.

We will abbreviate

$$\mathcal{H}_P^\vartheta[\beta_1, \dots, \beta_k] = \mathcal{H}_{h_0(P)}^\vartheta[h_1(P), \beta_1, \dots, \beta_k].$$

As in the previous section, the point of the following is that it can be extended beyond basic codes:

Lemma 5.7. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. The assignment of operators to basic $\mathbf{L}_{\omega^\alpha}^u$ -codes is locally correct in the sense that we have

- (H1) $k_{\langle \rangle}(P) \in \mathcal{H}_P^\vartheta$,
- (H2) $h_0(n(P, a)) \leq h_0(P)$,
- (H3) $\{h_1(n(P, a)), o_{\langle \rangle}(n(P, a))\} \subseteq \mathcal{H}_P^\vartheta[|a|]$

for any basic $\mathbf{L}_{\omega^\alpha}^u$ -code P and all $a \in \iota(r_{\langle \rangle}(P))$.

Proof. Write $P = P_\alpha^u \sigma$. To verify (H1) we have to distinguish two cases: Assume first that σ is a node in the preproof P_α^u . Then we have $o_{\langle \rangle}(P_\alpha^u \sigma) = o_\alpha^u(\sigma)$, where o_α^u is the labelling function of P_α^u . The same holds for the other labelling functions, so that we see $k_{\langle \rangle}(P_\alpha^u \sigma) = k_{P_\alpha^u}(\sigma)$. Using Lemma 5.4 we get

$$k_{\langle \rangle}(P_\alpha^u \sigma) \in \mathcal{H}_\Omega^\vartheta[|\sigma|] = \mathcal{H}_P^\vartheta.$$

If the node σ does not lie in P_α^u then we have $k_{\langle \rangle}(P_\alpha^u \sigma) = 0$ by default, so (H1) holds in this case as well. Condition (H2) is trivial for basic codes. As for (H3), recall that $n(P_\alpha^u \sigma, a)$ was defined to be the basic code $P_\alpha^u \sigma \frown a$. Thus we have

$$h_1(n(P_\alpha^u \sigma, a)) = |\sigma \frown a| = \max\{|\sigma|, |a|\} \in \mathcal{H}_\Omega^\vartheta[|\sigma|, |a|] = \mathcal{H}_{P_\alpha^u \sigma}^\vartheta[|a|].$$

Finally, condition (H1) for $n(P_\alpha^u \sigma, a) = P_\alpha^u \sigma \frown a$ gives

$$o_{\langle \rangle}(n(P_\alpha^u \sigma, a))^* \in \mathcal{H}_{P_\alpha^u \sigma \frown a}^\vartheta = \mathcal{H}_{P_\alpha^u \sigma}^\vartheta[|a|].$$

By Lemma 2.21 we get $o_{\langle \rangle}(n(P_\alpha^u \sigma, a)) \in \mathcal{H}_{P_\alpha^u \sigma}^\vartheta[|a|]$, as required for the second part of (H3). \square

Having seen the last part of this proof, the reader may wonder whether the condition $o_{\langle \rangle}(n(P, a)) \in \mathcal{H}_P^\vartheta[|a|]$ in (H3) is redundant. This is indeed the case once we have established conditions (H1-H3) for *all* $\mathbf{L}_{\omega^\alpha}^u$ -codes. However, in order to establish (H1-H3) by induction on the length of codes we will need condition (H3) as it stands. The following relies on the interpretation of codes as proofs (see Definition 4.4):

Corollary 5.8. Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For any basic $\mathbf{L}_{\omega^\alpha}^u$ -code P , the $\mathbf{L}_{\omega^\alpha}^u$ -preproof $[P]$ is controlled by the operator \mathcal{H}_P^ϑ .

Proof. Recall that we have $o_{[P]}(\sigma) = o_{\langle \rangle}(\bar{n}(P, \sigma))$ for any node $\sigma \in [P]$. The same holds for the other labels, so we get $k_{[P]}(\sigma) = k_{\langle \rangle}(\bar{n}(P, \sigma))$. By condition (H1) from the previous lemma we obtain

$$k_{[P]}(\sigma) \in \mathcal{H}_{\bar{n}(P, \sigma)}^{\vartheta}.$$

Thus it suffices to establish

$$\mathcal{H}_{\bar{n}(P, \sigma)}^{\vartheta} \subseteq \mathcal{H}_P^{\vartheta}[|\sigma|].$$

We prove the stronger claim

$$\mathcal{H}_{\bar{n}(P, \tau)}^{\vartheta}[|\sigma|] \subseteq \mathcal{H}_P^{\vartheta}[|\sigma|].$$

by induction on initial segments τ of σ : For $\tau = \langle \rangle$ we have $\bar{n}(P, \tau) = P$ and the claim is trivial. In the induction step, write $\tau = \rho \frown a$. In view of $\bar{n}(P, \tau) = n(\bar{n}(P, \rho), a)$ condition (H3) yields

$$h_1(\bar{n}(P, \tau)) \in \mathcal{H}_{\bar{n}(P, \rho)}^{\vartheta}[|a|].$$

Since $\tau = \rho \frown a$ is an initial segment of σ we have $|a| \leq |\sigma|$. Together with the induction hypothesis we obtain

$$h_1(\bar{n}(P, \tau)) \in \mathcal{H}_{\bar{n}(P, \rho)}^{\vartheta}[|\sigma|] \subseteq \mathcal{H}_P^{\vartheta}[|\sigma|].$$

By Lemma 2.24 this implies

$$\mathcal{H}_{h_0(P)}^{\vartheta}[h_1(\bar{n}(P, \tau)), |\sigma|] \subseteq \mathcal{H}_P^{\vartheta}[|\sigma|].$$

Iterative applications of (H2) yield $h_0(\bar{n}(P, \tau)) \leq h_0(P)$. Using Proposition 2.27 we can thus conclude

$$\mathcal{H}_{\bar{n}(P, \tau)}^{\vartheta}[|\sigma|] = \mathcal{H}_{h_0(\bar{n}(P, \tau))}^{\vartheta}[h_1(\bar{n}(P, \tau)), |\sigma|] \subseteq \mathcal{H}_{h_0(P)}^{\vartheta}[h_1(\bar{n}(P, \tau)), |\sigma|] \subseteq \mathcal{H}_P^{\vartheta}[|\sigma|],$$

as required for the induction step. \square

Similar to the treatment of cut-rank in the previous section, we can now extend the assignment of operators beyond basic codes: To do so it is enough to find an extension of h_0 and h_1 which remains locally correct in the sense of conditions (H1) to (H3). The (proof of the) corollary will guarantee that the assigned operators do indeed control the interpretations of the codes.

Lemma 5.9. *Assume $\vartheta : \varepsilon(S_{\omega}^u) \xrightarrow{\text{BH}} \alpha$. The operator assignment $\langle h_0, h_1 \rangle$ from Definition 5.6 can be extended to all \mathbf{L}_{ω}^u -codes constructed in the previous section. In particular this assignment gives $h_i(\mathcal{E}P) = h_i(P)$ for $i = 0, 1$ and any \mathbf{L}_{ω}^u -code P .*

Proof. We set $h_0(P) = \Omega$ for all codes P from the previous section (different values of $h_0(P)$ will occur later in this section). So condition (H2) from Lemma 5.7 is immediate. The value $h_1(P)$ is defined by recursion over the code P : Definition 5.6 accounts for the base case of a basic \mathbf{L}_{ω}^u -code. As a first recursive case, consider a term of the form $\mathcal{I}_{\forall_x \psi, b}P$. We put

$$h_1(\mathcal{I}_{\forall_x \psi, b}P) := \max\{h_1(P), k(\forall_x \psi), |b|\}.$$

This is designed to satisfy condition (H1): Observe

$$k(l_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)) = \max\{k(l_{\langle \rangle}(P)), k(\psi(b))\} \leq \max\{k_{\langle \rangle}(P), k(\forall_x \psi), |b|\}$$

and

$$k(r_{\langle \rangle}(\mathcal{I}_{\forall_x \psi, b}P)) = \max\{k(r_{\langle \rangle}(P)), |b|\} \leq \max\{k_{\langle \rangle}(P), |b|\},$$

where $|b|$ accounts for the possibility of a new rule $r_{\langle \rangle}(\mathcal{I}_{\forall x \psi, b}P) = (\text{Rep}, b)$. Together with $o_{\langle \rangle}(\mathcal{I}_{\forall x \psi, b}P) = o_{\langle \rangle}(P)$ this implies

$$k_{\langle \rangle}(\mathcal{I}_{\forall x \psi, b}P) \leq \max\{k_{\langle \rangle}(P), k(\forall x \psi), |b|\}.$$

To establish (H1) for $\mathcal{I}_{\forall x \psi, b}P$ it is thus enough to show

$$\{k_{\langle \rangle}(P), k(\forall x \psi), |b|\} \subseteq \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}.$$

By the induction hypothesis (condition (H1) for P) and $h_1(P) \leq h_1(\mathcal{I}_{\forall x \psi, b}P)$ we get

$$k_{\langle \rangle}(P) \in \mathcal{H}_P^{\vartheta} \subseteq \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}.$$

By definition of the operators we have $h_1(\mathcal{I}_{\forall x \psi, b}P) \in \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}$. Together with $\max\{k(\forall x \psi), |b|\} \leq h_1(\mathcal{I}_{\forall x \psi, b}P)$ this implies

$$\{k(\forall x \psi), |b|\} \subseteq \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta},$$

which completes the verification of (H1) for $\mathcal{I}_{\forall x \psi, b}P$. As for condition (H3), in view of $n(\mathcal{I}_{\forall x \psi, b}P, a) = \mathcal{I}_{\forall x \psi, b}n(P, a)$ we have

$$h_1(n(\mathcal{I}_{\forall x \psi, b}P, a)) = \max\{h_1(n(P, a)), k(\forall x \psi), |b|\}.$$

Observe $\iota(r_{\langle \rangle}(\mathcal{I}_{\forall x \psi, b}P)) \subseteq \iota(r_{\langle \rangle}(P))$. Thus condition (H3) for P gives

$$h_1(n(P, a)) \in \mathcal{H}_P^{\vartheta}[|a|] \subseteq \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}[|a|]$$

for all $a \in \iota(r_{\langle \rangle}(\mathcal{I}_{\forall x \psi, b}P))$. Together with the above we get

$$h_1(n(\mathcal{I}_{\forall x \psi, b}P, a)) \in \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}[|a|].$$

Similarly, we can infer

$$o_{\langle \rangle}(n(\mathcal{I}_{\forall x \psi, b}P, a)) = o_{\langle \rangle}(n(P, a)) \in \mathcal{H}_P^{\vartheta}[|a|] \subseteq \mathcal{H}_{\mathcal{I}_{\forall x \psi, b}P}^{\vartheta}[|a|]$$

from condition (H3) for P .

For terms of the form $\mathcal{I}_{\psi_0 \wedge \psi_1, i}P$ we set

$$h_1(\mathcal{I}_{\psi_0 \wedge \psi_1, i}P) := \max\{h_1(P), k(\psi_i)\}.$$

Conditions (H1) to (H3) are verified as above (we now have to accomodate a new rule (Rep, i) , but $i \in \mathcal{H}_t[\beta]$ is automatic for $i = 0, 1$).

Let us come to the case of a term $\mathcal{R}_{\exists x \psi}P_0P_1$: Here we put

$$h_1(\mathcal{R}_{\exists x \psi}P_0P_1) := \max\{h_1(P_0), h_1(P_1)\}.$$

As a preparation for (H1), let us show

$$k_{\langle \rangle}(\mathcal{R}_{\exists x \psi}P_0P_1) \leq \max\{k_{\langle \rangle}(P_0), k_{\langle \rangle}(P_1)\}.$$

Concerning the “ordinal” labels, we have

$$\begin{aligned} o_{\langle \rangle}(\mathcal{R}_{\exists x \psi}P_0P_1)^* &= (o_{\langle \rangle}(P_1) + o_{\langle \rangle}(P_0))^* \leq \\ &\leq \max\{o_{\langle \rangle}(P_0)^*, o_{\langle \rangle}(P_1)^*\} \leq \max\{k_{\langle \rangle}(P_0), k_{\langle \rangle}(P_1)\}. \end{aligned}$$

As for the end sequents, $l_{\langle \rangle}(\mathcal{R}_{\exists x \psi}P_0P_1) \subseteq l_{\langle \rangle}(P_0) \cup l_{\langle \rangle}(P_1)$ implies

$$k(l_{\langle \rangle}(\mathcal{R}_{\exists x \psi}P_0P_1)) \leq \max\{k(l_{\langle \rangle}(P_0)), k(l_{\langle \rangle}(P_1))\} \leq \max\{k_{\langle \rangle}(P_0), k_{\langle \rangle}(P_1)\}.$$

Concerning the last rule of the preproof $\mathcal{R}_{\exists x \psi}P_0P_1$, the only interesting case is $r_{\langle \rangle}(P_0) = (\exists, b, \psi)$ and $r_{\langle \rangle}(\mathcal{R}_{\exists x \psi}P_0P_1) = (\text{Cut}, \psi(b))$. Here we have

$$|b| \leq k(r_{\langle \rangle}(P_0)) \leq k_{\langle \rangle}(P_0).$$

Also, condition (L) for P_0 implies that the formula $\exists_x \psi$ occurs in $l_{\langle \rangle}(P_0)$. Thus we obtain

$$k(r_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1)) = k(\psi(b)) \leq \max\{k(\exists_x \psi), |b|\} \leq k_{\langle \rangle}(P_0).$$

Now that we know $k_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) \leq \max\{k_{\langle \rangle}(P_0), k_{\langle \rangle}(P_1)\}$ condition (H1) is easily established: From the induction hypothesis

$$k_{\langle \rangle}(P_i) \in \mathcal{H}_{P_i}^{\vartheta} \subseteq \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}$$

we can infer

$$k_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} P_0 P_1) \in \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}.$$

To verify condition (H3) we need to distinguish two cases: Assume first that we have $r_{\langle \rangle}(P_0) = (\exists_x, b, \psi)$ and $a = 1$. Then $n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a)$ was defined to be $\mathcal{I}_{\forall_x \neg \psi, b} P_1$. As observed above this implies $\max\{k(\forall_x \neg \psi), |b|\} \leq k_{\langle \rangle}(P_0)$. Thus we get

$$h_1(\mathcal{I}_{\forall_x \neg \psi, b} P_1) = \max\{h_1(P_1), k(\forall_x \neg \psi), |b|\} \leq \max\{h_1(P_1), k_{\langle \rangle}(P_0)\}.$$

Now $h_1(P_1) \in \mathcal{H}_{P_1}^{\vartheta}$ and $k_{\langle \rangle}(P_0) \in \mathcal{H}_{P_0}^{\vartheta}$ (condition (H1) for P_0) imply

$$h_1(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a)) = h_1(\mathcal{I}_{\forall_x \neg \psi, b} P_1) \in \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta} \subseteq \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}[[a]],$$

as required for (H3). Using condition (H1) for P_1 we also get

$$o_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a)) = o_{\langle \rangle}(\mathcal{I}_{\forall_x \neg \psi, b} P_1) = o_{\langle \rangle}(P_1) \in \mathcal{H}_{P_1}^{\vartheta} \subseteq \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}[[a]].$$

It remains to consider the case $n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a) = \mathcal{R}_{\exists_x \psi} n(P_0, a) P_1$. Here we have

$$h_1(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a)) = \max\{h_1(n(P_0, a)), h_1(P_1)\}.$$

We have $h_1(P_1) \in \mathcal{H}_{P_1}^{\vartheta} \subseteq \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}[[a]]$ and condition (H3) for P_0 yields

$$h_1(n(P_0, a)) \in \mathcal{H}_{P_0}^{\vartheta}[[a]] \subseteq \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}[[a]].$$

Finally, we have $o_{\langle \rangle}(P_1) \in \mathcal{H}_{P_1}^{\vartheta}$ and $o_{\langle \rangle}(n(P_0, a)) \in \mathcal{H}_{P_0}^{\vartheta}[[a]]$ from (H1) for P_1 resp. (H3) for P_0 . This implies

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{R}_{\exists_x \psi} P_0 P_1, a)) &= o_{\langle \rangle}(\mathcal{R}_{\exists_x \psi} n(P_0, a) P_1) = \\ &= o_{\langle \rangle}(P_1) + o_{\langle \rangle}(n(P_0, a)) \in \mathcal{H}_{\mathcal{R}_{\exists_x \psi} P_0 P_1}^{\vartheta}[[a]], \end{aligned}$$

as required by condition (H3) for $\mathcal{R}_{\exists_x \psi} P_0 P_1$.

The last remaining case is that of a term $\mathcal{E}P$: As announced in the statement of the lemma we set

$$h_1(\mathcal{E}P) := h_1(P).$$

Observing $(\Omega^s)^* \leq s^*$ it is easy to see

$$k_{\langle \rangle}(\mathcal{E}P) \leq k_{\langle \rangle}(P).$$

Thus condition (H1) for P implies

$$k_{\langle \rangle}(\mathcal{E}P) \in \mathcal{H}_P^{\vartheta} = \mathcal{H}_{\mathcal{E}P}^{\vartheta},$$

as required by condition (H1) for $\mathcal{E}P$. Concerning (H3), let us first assume that $r_{\langle \rangle}(P)$ is not a cut, or a cut over a formula of height at most one. Then we have $n(\mathcal{E}P, a) = \mathcal{E}n(P, a)$. Then condition (H3) for P yields

$$h_1(n(\mathcal{E}P, a)) = h_1(\mathcal{E}n(P, a)) = h_1(n(P, a)) \in \mathcal{H}_P^{\vartheta}[[a]] = \mathcal{H}_{\mathcal{E}P}^{\vartheta}[[a]],$$

as required by condition (H3) for $\mathcal{E}P$. Also, (H3) for P gives $o_{\langle \rangle}(n(P, a)) \in \mathcal{H}_P^{\vartheta}[[a]]$.

Thus we get

$$o_{\langle \rangle}(n(\mathcal{E}P, a)) = \Omega^{o_{\langle \rangle}(n(P, a))} \in \mathcal{H}_{\mathcal{E}P}^{\vartheta}[[a]].$$

Next, consider the case $r_{\langle \rangle}(P) = (\text{Cut}, \exists_x \psi)$ with $\text{ht}(\exists_x \psi) > 1$. Here we have

$$n(\mathcal{E}P, a) = \mathcal{R}_{\exists_x \psi}(\mathcal{E}n(P, 0))(\mathcal{E}n(P, 1)).$$

Condition (H3) for P gives

$$h_1(n(P, i)) \in \mathcal{H}_P^\vartheta[|i|] = \mathcal{H}_{\mathcal{E}P}^\vartheta.$$

Thus we get

$$h_1(n(\mathcal{E}P, a)) = \max\{h_1(n(P, 0)), h_1(n(P, 1))\} \in \mathcal{H}_{\mathcal{E}P}^\vartheta[|a|].$$

Also, $o_{\langle \rangle}(n(P, i)) \in \mathcal{H}_P^\vartheta[|i|] = \mathcal{H}_{\mathcal{E}P}^\vartheta$ implies

$$o_{\langle \rangle}(n(\mathcal{E}P, a)) = \Omega^{o_{\langle \rangle}(n(P, 1))} + \Omega^{o_{\langle \rangle}(n(P, 0))} \in \mathcal{H}_{\mathcal{E}P}^\vartheta[|a|],$$

as required by condition (H3) for $\mathcal{E}P$. The remaining cases (cut-formulas of the forms $\forall_x \psi$, $\psi_0 \vee \psi_1$ and $\psi_0 \wedge \psi_1$) are similar and left to the reader. \square

In particular we obtain operator control for the preproof $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ from Proposition 4.14. Recall that this proof had empty end-sequent and cut rank 2 (but height above Ω).

Corollary 5.10. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. For any $C \in \omega$ the $\mathbf{L}_{\omega^\alpha}^u$ -preproof $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ is controlled by the operator $\mathcal{H}_\Omega^\vartheta$.*

Proof. The previous lemma and Definition 5.6 give $h_0(\mathcal{E}^C P_\alpha^u \langle \rangle) = h_0(P_\alpha^u \langle \rangle) = \Omega$ and $h_1(\mathcal{E}^C P_\alpha^u \langle \rangle) = h_1(P_\alpha^u \langle \rangle) = |\langle \rangle| = 0$. By Corollary 5.8 this means that $[\mathcal{E}^C P_\alpha^u \langle \rangle]$ is controlled by the operator $\mathcal{H}_\Omega^\vartheta[0] = \mathcal{H}_\Omega^\vartheta$. \square

In the first part of this section we have defined the notion of operator controlled $\mathbf{L}_{\omega^\alpha}^u$ -preproof. We have seen how this notion can be formulated in terms of $\mathbf{L}_{\omega^\alpha}^u$ -codes. Also, we have established operator control for all $\mathbf{L}_{\omega^\alpha}^u$ -codes constructed so far. Building on this, we can now move towards the collapsing procedure for proofs. Let us introduce some terminology: Consider an $\mathbf{L}_{\omega^\alpha}^u$ -formula φ and an ordinal $\beta < \omega^\alpha$. We write φ^β for the formula that results from φ when we replace each unbounded quantifier $\forall_x \dots$ resp. $\exists_x \dots$ by the bounded quantifier $\forall_{x \in \mathbb{L}_\beta^u} \dots$ resp. $\exists_{x \in \mathbb{L}_\beta^u} \dots$. In view of $|\mathbb{L}_\beta^u| = \beta$ we can conceive \mathbb{L}_β^u as an element of $\mathbf{L}_{\omega^\alpha}^u$, and φ^β as an $\mathbf{L}_{\omega^\alpha}^u$ -formula. Note that we have

$$k(\varphi^\beta) \leq \max\{k(\varphi), |\mathbb{L}_\beta^u|\} = \max\{k(\varphi), \beta\}.$$

A $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formula is an $\mathbf{L}_{\omega^\alpha}^u$ -formula which contains no unbounded universal quantifier. A $\Pi_1(\mathbf{L}_{\omega^\alpha}^u)$ -formula is an $\mathbf{L}_{\omega^\alpha}^u$ -formula of the form $\forall_x \theta$ where θ is bounded. The following boundedness result is a final preparation for collapsing:

Lemma 5.11. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. We can extend the system of $\mathbf{L}_{\omega^\alpha}^u$ -codes in the following way:*

- (a) *For each $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formula φ and each ordinal $\beta < \omega^\alpha$ we add a unary function symbol $\mathcal{B}_{\exists, \varphi}^\beta$ such that we have*

$$l_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) = (l_{\langle \rangle}(P) \setminus \{\varphi\}) \cup \{\varphi^\beta\},$$

$$o_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) = o_{\langle \rangle}(P),$$

$$d(\mathcal{B}_{\exists, \varphi}^\beta P) = d(P),$$

$$h_0(\mathcal{B}_{\exists, \varphi}^\beta P) = h_0(P),$$

$$h_1(\mathcal{B}_{\exists, \varphi}^\beta P) = \max\{h_1(P), k(\varphi^\beta)\}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -code P with $o_\langle \rangle(P) \leq \beta$.

- (b) For each unbounded $\Pi_1(\mathbf{L}_{\omega^\alpha}^u)$ -formula ψ and each ordinal $\beta < \omega^\alpha$ we add a unary function symbol $\mathcal{B}_{\forall, \psi}^\beta$ such that we have

$$l_\langle \rangle(\mathcal{B}_{\forall, \psi}^\beta P) = (l_\langle \rangle(P) \setminus \{\psi\}) \cup \{\psi^\beta\},$$

$$o_\langle \rangle(\mathcal{B}_{\forall, \psi}^\beta P) = o_\langle \rangle(P),$$

$$d(\mathcal{B}_{\forall, \psi}^\beta P) = d(P),$$

$$h_0(\mathcal{B}_{\forall, \psi}^\beta P) = h_0(P),$$

$$h_1(\mathcal{B}_{\forall, \psi}^\beta P) = \max\{h_1(P), k(\psi^\beta)\}$$

for any $\mathbf{L}_{\omega^\alpha}^u$ -code P (without any restriction on $o_\langle \rangle(P)$).

Proof. (a) Let us first say what happens in the unintended case $\beta < o_\langle \rangle(P)$: There we stipulate that $\mathcal{B}_{\exists_y \theta}^\beta P$ behaves as P , i.e. we set

$$\begin{aligned} l_\langle \rangle(\mathcal{B}_{\exists_y \theta}^\beta P) &= l_\langle \rangle(P), & o_\langle \rangle(\mathcal{B}_{\exists_y \theta}^\beta P) &= o_\langle \rangle(P), & r_\langle \rangle(\mathcal{B}_{\exists_y \theta}^\beta P) &= r_\langle \rangle(P), \\ n(\mathcal{B}_{\exists_y \theta}^\beta P, a) &= n(P, a), & d(\mathcal{B}_{\exists_y \theta}^\beta P) &= d(P), & h_i(\mathcal{B}_{\exists_y \theta}^\beta P) &= h_i(P). \end{aligned}$$

Then conditions (L), (C1,C2) and (H1-H3) for $\mathcal{B}_{\exists_y \theta}^\beta P$ immediately follow from the same conditions for P . In the intended case $o_\langle \rangle(P) \leq \beta$ it remains to extend the functions $r_\langle \rangle$ and n to terms of the form $\mathcal{B}_{\exists_y \theta}^\beta P$. We do this by case distinction on the rule $r_\langle \rangle(P)$, checking local correctness as we go along:

Case $r_\langle \rangle(P) = (\exists_x, b, \theta)$ with $\exists_x \theta \equiv \varphi$ and the outer quantifier unbounded: Intuitively, P deduces $\exists_x \theta$ from $\theta(b)$. Using the induction hypothesis we will be able to obtain $\theta(b)^\beta$. Also, condition (L) for P gives $|b| < o_\langle \rangle(P) \leq \beta$. Thus the bounded formula $b \in \mathbb{L}_\beta^u$ is true, and hence an axiom. We can introduce the conjunction $b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta$ and then the existential statement $\exists_{x \in \mathbb{L}_\beta^u} \theta^\beta \equiv \varphi^\beta$. To cast this in terms of codes we define a new constant $\mathbf{L}_{\omega^\alpha}^u$ -code Ax_θ for each true bounded $\mathbf{L}_{\omega^\alpha}^u$ -formula θ , setting

$$\begin{aligned} l_\langle \rangle(Ax_\theta) &= \langle \theta \rangle, & o_\langle \rangle(Ax_\theta) &= 0, & r_\langle \rangle(Ax_\theta) &= Ax, \\ n(Ax_\theta, a) &= Ax_\theta, & d(Ax_\theta) &= 0, & \langle h_0(Ax_\theta), h_1(Ax_\theta) \rangle &= \langle \Omega, k(\theta) \rangle. \end{aligned}$$

Conditions (L), (C1,C2) and (H1-H3) are easily checked. Also, we introduce a binary function symbol $\bigwedge_{\psi_0, \psi_1}$ on $\mathbf{L}_{\omega^\alpha}^u$ -codes, for all $\mathbf{L}_{\omega^\alpha}^u$ -formulas ψ_0, ψ_1 . This will serve to introduce conjunctions:

$$\begin{aligned} l_\langle \rangle(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= l_\langle \rangle(P_0) \setminus \{\psi_0\} \cup l_\langle \rangle(P_1) \setminus \{\psi_1\} \cup \{\psi_0 \wedge \psi_1\}, \\ o_\langle \rangle(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= \max\{o_\langle \rangle(P_0), o_\langle \rangle(P_1)\} + 1, \\ r_\langle \rangle(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= (\wedge, \psi_0, \psi_1), \\ n(\bigwedge_{\psi_0, \psi_1} P_0 P_1, a) &= \begin{cases} P_0 & \text{if } a = 0, \\ P_1 & \text{otherwise,} \end{cases} \\ d(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= \max\{d(P_0), d(P_1)\}, \\ h_0(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= \max\{h_0(P_0), h_0(P_1)\}, \\ h_1(\bigwedge_{\psi_0, \psi_1} P_0 P_1) &= \max\{h_1(P_0), h_1(P_1), k(\psi_0 \wedge \psi_1)\}. \end{aligned}$$

Again, it is straightforward to check conditions (L), (C1,C2) and (H1-H3). After these preparations we can address the code $\mathcal{B}_{\exists,\varphi}^\beta P$ itself: Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P) &= (\exists x, b, x \in \mathbb{L}_\beta^u \wedge \theta^\beta), \\ n(\mathcal{B}_{\exists,\varphi}^\beta P, a) &= \bigwedge_{b \in \mathbb{L}_\beta^u, \theta(b)^\beta} \text{Ax}_{b \in \mathbb{L}_\beta^u}(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0)). \end{aligned}$$

We have already observed $b \in \mathbb{L}_\beta^u$, which ensures that $\text{Ax}_{b \in \mathbb{L}_\beta^u}$ is an $\mathbf{L}_{\omega^\alpha}^u$ -code. Also note that $\theta(b)$ is a $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formula, so that $\mathcal{B}_{\exists,\theta(b)}^\beta$ is one of our new function symbols. As a preparation for local correctness, observe that condition (L) for P implies $o_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta n(P, 0)) = o_{\langle \rangle}(n(P, 0)) \leq o_{\langle \rangle}(P) \leq \beta$. Thus the codes $\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0)$ and $\mathcal{B}_{\exists,\varphi}^\beta n(P, 0)$ fall under the “intended case”. Now condition (L) for $\mathcal{B}_{\exists,\varphi}^\beta P$ holds by

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{B}_{\exists,\varphi}^\beta P, 0)) + \omega &= o_{\langle \rangle}(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0)) + 1 + \omega = \\ &= o_{\langle \rangle}(n(P, 0)) + 1 + \omega = o_{\langle \rangle}(n(P, 0)) + \omega \leq o_{\langle \rangle}(P) = o(\mathcal{B}_{\exists,\varphi}^\beta P) \end{aligned}$$

and

$$|b| < o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P),$$

as well as $\exists x \in \mathbb{L}_\beta^u \theta^\beta \equiv \varphi^\beta \in l_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P)$ (by definition) and

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{B}_{\exists,\varphi}^\beta P, 0)) &= l_{\langle \rangle}(\bigwedge_{b \in \mathbb{L}_\beta^u, \theta(b)^\beta} \text{Ax}_{b \in \mathbb{L}_\beta^u}(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0))) = \\ &= l_{\langle \rangle}(\text{Ax}_{b \in \mathbb{L}_\beta^u} \setminus \{b \in \mathbb{L}_\beta^u\} \cup l_{\langle \rangle}(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0)) \setminus \{\theta(b)^\beta\} \cup \{b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta\}) \subseteq \\ &\subseteq [l_{\langle \rangle}(n(P, 0)) \setminus \{\varphi, \theta(b)\} \cup \{\varphi^\beta, \theta(b)^\beta\}] \setminus \{\theta(b)^\beta\} \cup \{b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta\} \subseteq \\ &\subseteq l_{\langle \rangle}(P) \setminus \{\varphi\} \cup \{\varphi^\beta, b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta\} = l_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P) \cup \{b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta\}. \end{aligned}$$

Condition (C1) is void and condition (C2) follows from same condition for P . Concerning condition (H1), we have

$$k_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P) \leq \max\{k_{\langle \rangle}(P), k(\varphi^\beta)\} \leq \max\{k_{\langle \rangle}(P), h_1(\mathcal{B}_{\exists,\varphi}^\beta P)\}.$$

By (H1) for P we have $k_{\langle \rangle}(P) \in \mathcal{H}_P^\varphi \subseteq \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta$. Also, $h_1(\mathcal{B}_{\exists,\varphi}^\beta P) \in \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta$ follows from the definition of our operators. Together we obtain

$$k_{\langle \rangle}(\mathcal{B}_{\exists,\varphi}^\beta P) \in \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta,$$

as required by condition (H1) for $\mathcal{B}_{\exists,\varphi}^\beta P$. Condition (H2) is easily reduced to the same condition for P . Finally, let us verify condition (H3): From condition (L) for P we know that φ occurs in $l_{\langle \rangle}(P)$, so that we have $k(\theta) = k(\varphi) \leq k_{\langle \rangle}(P)$. Together with $|b| \leq k(r_{\langle \rangle}(P)) \leq k_{\langle \rangle}(P)$ we get

$$\begin{aligned} h_1(n(\mathcal{B}_{\exists,\varphi}^\beta P, 0)) &= h_1(\bigwedge_{b \in \mathbb{L}_\beta^u, \theta(b)^\beta} \text{Ax}_{b \in \mathbb{L}_\beta^u}(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0))) \leq \\ &\leq \max\{h_1(\mathcal{B}_{\exists,\theta(b)}^\beta \mathcal{B}_{\exists,\varphi}^\beta n(P, 0)), k(b \in \mathbb{L}_\beta^u \wedge \theta(b)^\beta)\} \leq \\ &\leq \max\{h_1(n(P, 0)), k(\varphi), |b|\} \leq \max\{h_1(n(P, 0)), h_1(\mathcal{B}_{\exists,\varphi}^\beta P), k_{\langle \rangle}(P)\}. \end{aligned}$$

Now $h_1(n(P, 0)) \in \mathcal{H}_P^\vartheta[[0]] \subseteq \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta[[0]]$ holds by (H3) for P ; by definition of our operators we have $h_1(\mathcal{B}_{\exists,\varphi}^\beta P) \in \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta[[0]]$; and $k_{\langle \rangle}(P) \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_{\mathcal{B}_{\exists,\varphi}^\beta P}^\vartheta[[0]]$ is due

to (H1) for P . Together this gives

$$h_1(n(\mathcal{B}_{\exists, \varphi}^\beta P, 0)) \in \mathcal{H}_{\mathcal{B}_{\exists, \varphi}^\beta P}^\vartheta[[0]],$$

as required as required by condition (H3) for $\mathcal{B}_{\exists, \varphi}^\beta P$. Still concerning condition (H3), we have

$$o_{\langle \rangle}(n(\mathcal{B}_{\exists, \varphi}^\beta P, 0)) = o_{\langle \rangle}(\mathcal{B}_{\exists, \theta(b)}^\beta \mathcal{B}_{\exists, \varphi}^\beta n(P, 0)) + 1 = o_{\langle \rangle}(n(P, 0)) + 1.$$

Condition (H3) for P provides $o_{\langle \rangle}(n(P, 0)) \in \mathcal{H}_P^\vartheta[[0]] \subseteq \mathcal{H}_{\mathcal{B}_{\exists, \varphi}^\beta P}^\vartheta[[0]]$, which implies

$$o_{\langle \rangle}(n(\mathcal{B}_{\exists, \varphi}^\beta P, 0)) \in \mathcal{H}_{\mathcal{B}_{\exists, \varphi}^\beta P}^\vartheta[[0]],$$

as required by condition (H3) for $\mathcal{B}_{\exists, \varphi}^\beta P$.

Case $r_{\langle \rangle}(P) = (\exists_x, b, \theta)$ with $\exists_x \theta \equiv \varphi$ and the outer quantifier bounded: Note that we now have $\exists_x \theta^\beta \equiv \varphi^\beta$. We set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) &= (\exists_x, b, \theta^\beta), \\ n(\mathcal{B}_{\exists, \varphi}^\beta P, a) &= \mathcal{B}_{\exists, \theta(b)}^\beta \mathcal{B}_{\exists, \varphi}^\beta n(P, 0). \end{aligned}$$

The verification of local correctness is easier than in the previous case.

Case $r_{\langle \rangle}(P) = (\exists_x, b, \theta)$ with $\exists_x \theta \not\equiv \varphi$: Here we put

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) &= r_{\langle \rangle}(P), \\ n(\mathcal{B}_{\exists, \varphi}^\beta P, a) &= \mathcal{B}_{\exists, \varphi}^\beta n(P, a). \end{aligned}$$

The verification of local correctness is easier than in the previous cases.

The rule $r_{\langle \rangle}(P) = (\wedge, \psi_0, \psi_1)$ is treated similarly, distinguishing the two cases $\varphi \equiv \psi_0 \wedge \psi_1$ and $\varphi \not\equiv \psi_0 \wedge \psi_1$. The same applies to the rules $r_{\langle \rangle}(P) = (\vee_i, \psi_0, \psi_1)$ and $r_{\langle \rangle}(P) = (\forall_x, \psi)$ (if $\forall_x \psi \equiv \varphi$ then the outer quantifier must be bounded, which implies $\forall_x(\psi^\beta) \equiv \varphi^\beta$). Let us look at the remaining cases:

Case $r_{\langle \rangle}(P) = \text{Ax}$: We set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) &= \text{Ax}, \\ n(\mathcal{B}_{\exists, \varphi}^\beta P, a) &= \mathcal{B}_{\exists, \varphi}^\beta n(P, a). \end{aligned}$$

By condition (L) for P the sequent $l_{\langle \rangle}(P)$ contains a true bounded formula. The same formula is still contained in $l_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) = (l_{\langle \rangle}(P) \setminus \{\varphi\}) \cup \{\varphi^\beta\}$, as we have $\varphi^\beta \equiv \varphi$ if φ is bounded. The remaining conditions are verified as above.

Case $r_{\langle \rangle}(P) = (\text{Cut}, \psi)$: Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) &= (\text{Cut}, \psi), \\ n(\mathcal{B}_{\exists, \varphi}^\beta P, a) &= \mathcal{B}_{\exists, \varphi}^\beta n(P, a). \end{aligned}$$

The verification of local correctness is straightforward.

Case $r_{\langle \rangle}(P) = (\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta)$: By condition (L) for P this would require $\Omega \leq o_{\langle \rangle}(P) \leq \beta$, which is incompatible with the assumption $\beta < \omega^\alpha$.

Case $r_{\langle \rangle}(P) = (\text{Rep}, b)$: Set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\exists, \varphi}^\beta P) &= (\text{Rep}, b), \\ n(\mathcal{B}_{\exists, \varphi}^\beta P, a) &= \mathcal{B}_{\exists, \varphi}^\beta n(P, a). \end{aligned}$$

The verification of local correctness is straightforward.

(b) The proof is similar to that of part (a), but the assumption that ψ is an unbounded $\Pi_1(\mathbf{L}_{\omega^\alpha}^u)$ -formula saves us some cases: For example it implies that ψ cannot be of the form $\psi_0 \wedge \psi_1$. We write out the only interesting case, leaving the other cases to the reader:

Case $r_{\langle \rangle}(P) = (\forall_x, \theta)$ with $\forall_x \theta \equiv \psi$ and the outer quantifier unbounded: Intuitively, P deduces $\forall_x \theta$ from the assumptions $\theta(a)$ for all $a \in \mathbf{L}_{\omega^\alpha}^u$. Introducing disjunctions we get $a \notin \mathbb{L}_\beta^u \vee \theta(a)$, from which we obtain $\forall_{x \in \mathbb{L}_\beta^u} \theta \equiv \psi$. To formulate this in terms of codes we introduce a unary function symbol $\bigvee_{\psi_0, \psi_1}^i$ for $i = 0, 1$ and arbitrary $\mathbf{L}_{\omega^\alpha}^u$ -formulas ψ_0, ψ_1 . This serves to introduce disjunctions:

$$\begin{aligned} l_{\langle \rangle}(\bigvee_{\psi_0, \psi_1}^i P) &= l_{\langle \rangle}(P) \setminus \{\psi_i\} \cup \{\psi_0 \vee \psi_1\}, \\ o_{\langle \rangle}(\bigvee_{\psi_0, \psi_1}^i P) &= o_{\langle \rangle}(P) + 1, \\ r_{\langle \rangle}(\bigvee_{\psi_0, \psi_1}^i P) &= (\forall_i, \psi_0, \psi_1), \\ n(\bigvee_{\psi_0, \psi_1}^i P, a) &= P, \\ d(\bigvee_{\psi_0, \psi_1}^i P) &= d(P), \\ h_0(\bigvee_{\psi_0, \psi_1}^i P) &= h_0(P), \\ h_1(\bigvee_{\psi_0, \psi_1}^i P) &= \max\{h_1(P), k(\psi_0 \vee \psi_1)\}. \end{aligned}$$

It is straightforward to check local correctness. Now put

$$\begin{aligned} r_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta P) &= (\forall_x, x \notin \mathbb{L}_\beta^u \vee \theta), \\ n(\mathcal{B}_{\forall, \psi}^\beta P, a) &= \bigvee_{a \notin \mathbb{L}_\beta^u, \theta(a)}^1 \mathcal{B}_{\forall, \psi}^\beta n(P, a). \end{aligned}$$

Let us verify condition (L) for $\mathcal{B}_{\forall, \psi}^\beta P$: Concerning the “ordinal” labels we have

$$\begin{aligned} o_{\langle \rangle}(n(\mathcal{B}_{\forall, \psi}^\beta P, a)) + \omega &= o_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta n(P, a)) + 1 + \omega = o_{\langle \rangle}(n(P, a)) + 1 + \omega = \\ &= o_{\langle \rangle}(n(P, a)) + \omega \leq o_{\langle \rangle}(P) = o_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta P). \end{aligned}$$

The formula $\forall_{x \in \mathbb{L}_\beta^u} \theta \equiv \varphi^\beta$ occurs in $l_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta P)$ by definition. We also have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{B}_{\forall, \psi}^\beta P, a)) &= l_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta n(P, a)) \setminus \{\theta(a)\} \cup \{a \notin \mathbb{L}_\beta^u \vee \theta(a)\} \subseteq \\ &\subseteq l_{\langle \rangle}(n(P, a)) \setminus \{\varphi, \theta(a)\} \cup \{\varphi^\beta, a \notin \mathbb{L}_\beta^u \vee \theta(a)\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P) \cup \{\theta(a)\}) \setminus \{\varphi, \theta(a)\} \cup \{\varphi^\beta, a \notin \mathbb{L}_\beta^u \vee \theta(a)\} \subseteq \\ &\subseteq l_{\langle \rangle}(P) \setminus \{\varphi\} \cup \{\varphi^\beta, a \notin \mathbb{L}_\beta^u \vee \theta(a)\} = l_{\langle \rangle}(n(\mathcal{B}_{\forall, \psi}^\beta P) \cup \{a \notin \mathbb{L}_\beta^u \vee \theta(a)\}), \end{aligned}$$

as required by condition (L) at the rule $(\forall_x, x \notin \mathbb{L}_\beta^u \vee \theta)$. Condition (C1) does not apply, and (C2) is easily reduced to the same condition for P . As for (H1), similar to part (a) one sees

$$k_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta P) \leq \max\{k_{\langle \rangle}(P), k(\psi_0 \vee \psi_1)\} \leq \max\{k_{\langle \rangle}(P), h_1(\mathcal{B}_{\forall, \psi}^\beta P)\}.$$

By (H1) for P we have $k_{\langle \rangle}(P) \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_{\mathcal{B}_{\forall, \psi}^\beta P}^\vartheta$. Also, $h_1(\mathcal{B}_{\forall, \psi}^\beta P) \in \mathcal{H}_{\mathcal{B}_{\forall, \psi}^\beta P}^\vartheta$ holds by the definition of our operators. Together we obtain

$$k_{\langle \rangle}(\mathcal{B}_{\forall, \psi}^\beta P) \in \mathcal{H}_{\mathcal{B}_{\forall, \psi}^\beta P}^\vartheta,$$

as required by condition (H1) for $\mathcal{B}_{\vartheta, \psi}^\beta P$. Condition (H2) is easily reduced to the same condition for P . As for (H3), it is straightforward to see

$$h_1(n(\mathcal{B}_{\vartheta, \psi}^\beta P, a)) \leq \max\{h_1(n(P, a)), k(\psi^\beta), |a|\} \leq \max\{h_1(n(P, a)), h_1(\mathcal{B}_{\vartheta, \psi}^\beta P), |a|\}.$$

Condition (H3) for P gives $h_1(n(P, a)) \in \mathcal{H}_P^\vartheta[|a|] \subseteq \mathcal{H}_{\mathcal{B}_{\vartheta, \psi}^\beta P}^\vartheta[|a|]$. Also, the definition of our operators yields $\{h_1(\mathcal{B}_{\vartheta, \psi}^\beta P), |a|\} \subseteq \mathcal{H}_{\mathcal{B}_{\vartheta, \psi}^\beta P}^\vartheta[|a|]$. Together we get

$$h_1(n(\mathcal{B}_{\vartheta, \psi}^\beta P, a)) \in \mathcal{H}_{\mathcal{B}_{\vartheta, \psi}^\beta P}^\vartheta[|a|],$$

as required by condition (H3) for $\mathcal{B}_{\vartheta, \psi}^\beta P$. Still concerning (H3), we have

$$o_{\langle \rangle}(n(\mathcal{B}_{\vartheta, \psi}^\beta P, a)) = o_{\langle \rangle}(n(P, a)) + 1.$$

Condition (H3) for P gives $o_{\langle \rangle}(n(P, a)) \in \mathcal{H}_P^\vartheta[|a|] \subseteq \mathcal{H}_{\mathcal{B}_{\vartheta, \psi}^\beta P}^\vartheta[|a|]$. This implies

$$o_{\langle \rangle}(n(\mathcal{B}_{\vartheta, \psi}^\beta P, a)) \in \mathcal{H}_{\mathcal{B}_{\vartheta, \psi}^\beta P}^\vartheta[|a|],$$

as required by condition (H3) for $\mathcal{B}_{\vartheta, \psi}^\beta P$. \square

The last two sections have been a preparation of the following collapsing result:

Theorem 5.12. *Assume $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$. We can extend the system of $\mathbf{L}_{\omega^\alpha}^u$ -codes by a unary function symbol \mathcal{C}_t for each $t \in \varepsilon(S_{\omega^\alpha}^u)$ with $\Omega \leq t$, in such a way that we have*

$$\begin{aligned} l_{\langle \rangle}(\mathcal{C}_t P) &= l_{\langle \rangle}(P), & o_{\langle \rangle}(\mathcal{C}_t P) &= \vartheta(t + \Omega^{o_{\langle \rangle}(P)}), \\ d(\mathcal{C}_t P) &= 1, & \langle h_0(\mathcal{C}_t P), h_1(\mathcal{C}_t P) \rangle &= \langle t + \Omega^{o_{\langle \rangle}(P)}, 0 \rangle \end{aligned}$$

whenever the following conditions are satisfied:

- (i) $l_{\langle \rangle}(P)$ contains only $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formulas and $d(P) \leq 2$,
- (ii) $h_0(P) \leq t$ and $\{t, h_1(P)\} \subseteq \mathcal{H}_t^\vartheta$.

Proof. First, in the unintended case where one of the conditions (i,ii) fails we stipulate that $\mathcal{C}_t P$ behaves like P (see the previous proof). In this case, the local correctness of $\mathcal{C}_t P$ follows from the local correctness of P . Now assume that conditions (i) and (ii) hold for P . We define $r_{\langle \rangle}(\mathcal{C}_t P)$ and $n(\mathcal{C}_t P, a)$ by case distinction on $r_{\langle \rangle}(P)$, verifying local correctness as we go along:

Case $r_{\langle \rangle}(P) = \text{Ax}$: We set $r_{\langle \rangle}(\mathcal{C}_t P) = \text{Ax}$ and $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. Condition (L) for P implies that $l_{\langle \rangle}(P)$ contains a true bounded formula. Thus the same holds for $l_{\langle \rangle}(\mathcal{C}_t P) = l_{\langle \rangle}(P)$, as required by condition (L) for $\mathcal{C}_t(P)$. Condition (C1) is trivial as $r_{\langle \rangle}(\mathcal{C}_t P)$ is not a cut rule. Condition (C2) does not apply as $\iota(\text{Ax}) = \emptyset$. Concerning condition (H1), it is easy to see

$$k_{\langle \rangle}(\mathcal{C}_t P) \leq \max\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(P)})^*\}.$$

By condition (H1) for P and assumption (ii) we get

$$k_{\langle \rangle}(P) \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta[h_1(P)] = \mathcal{H}_t^\vartheta \subseteq \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^\vartheta = \mathcal{H}_{\mathcal{C}_t P}^\vartheta.$$

In particular we have $o_{\langle \rangle}(P) \in \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^\vartheta$. Together with assumption (ii) we obtain

$$\vartheta(t + \Omega^{o_{\langle \rangle}(P)}) \in \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^\vartheta = \mathcal{H}_{\mathcal{C}_t P}^\vartheta$$

and then $\vartheta(t + \Omega^{\circ \diamond}(P))^* \in \mathcal{H}_{\mathcal{C}_t P}^\vartheta$, completing the proof of condition (H1) for $\mathcal{C}_t P$. Conditions (H2) and (H3) do not apply, because of $\iota(\text{Ax}) = \emptyset$.

Case $r_\diamond(P) = (\wedge, \psi_0, \psi_1)$: Set $r_\diamond(\mathcal{C}_t P) = (\wedge, \psi_0, \psi_1)$ and $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. Before we can verify local correctness we must check that $n(P, i)$ satisfies assumptions (i,ii) for $i = 0, 1$: By condition (L) for P the formula $\psi_0 \wedge \psi_1$ occurs in $l_\diamond(P)$. Thus assumption (i) for P implies that $\psi_0 \wedge \psi_1$ is a $\Sigma(\mathbf{L}_{\omega^\alpha})$ -formula, and so are ψ_0 and ψ_1 . Also by condition (L) for P we have

$$l_\diamond(n(P, i)) \subseteq l_\diamond(P) \cup \{\psi_i\}.$$

Thus $l_\diamond(n(P, i))$ consists of $\Sigma(\mathbf{L}_{\omega^\alpha})$ -formulas. Next, by condition (C2) and assumption (i) for P we have $d(n(P, i)) \leq d(P) \leq 2$. So $n(P, i)$ satisfies assumption (i). Concerning assumption (ii), using (H2) for P we obtain $h_0(n(P, i)) \leq h_0(P) \leq t$. Also, condition (H3) and assumption (ii) for P yield

$$h_1(n(P, i)) \in \mathcal{H}_P^\vartheta[[i]] = \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta.$$

We have established that $n(P, i)$ satisfies assumptions (i,ii). So $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$ falls under the “intended case”. Now we can verify condition (L) for $\mathcal{C}_t P$: By condition (H3) and assumption (ii) for P we get

$$o_\diamond(n(P, i)) \in \mathcal{H}_P^\vartheta[[i]] = \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta.$$

Assumption (ii) provides $t \in \mathcal{H}_t^\vartheta$ and condition (L) for P gives $o_\diamond(n(P, i)) < o_\diamond(P)$. In this situation Proposition 2.27 yields

$$o_\diamond(n(\mathcal{C}_t P, i)) = \vartheta(t + \Omega^{\circ \diamond}(n(P, i))) < \vartheta(t + \Omega^{\circ \diamond}(P)) = o_\diamond(\mathcal{C}_t P),$$

as required by condition (L) for $\mathcal{C}_t P$. Still concerning condition (L) for $\mathcal{C}_t P$, by the local correctness of P the formula $\psi_0 \wedge \psi_1$ occurs in $l_\diamond(P) = l_\diamond(\mathcal{C}_t P)$. Also we have

$$l_\diamond(n(\mathcal{C}_t P, i)) = l_\diamond(n(P, i)) \subseteq l_\diamond(P) \cup \{\psi_i\} = l_\diamond(\mathcal{C}_t P) \cup \{\psi_i\},$$

completing the verification of (L) for $\mathcal{C}_t P$. Condition (H1) is established as in the case of an axiom above. In view of $o_\diamond(n(P, i)) < o_\diamond(P)$ we get

$$h_0(n(\mathcal{C}_t P, i)) = t + \Omega^{\circ \diamond}(n(P, i)) < t + \Omega^{\circ \diamond}(P) = h_0(\mathcal{C}_t P),$$

which settles condition (H2) for $\mathcal{C}_t P$. The first part of (H3) is automatic by

$$h_1(n(\mathcal{C}_t P, i)) = h_1(\mathcal{C}_t n(P, i)) = 0 \in \mathcal{H}_{\mathcal{C}_t P}^\vartheta[[i]].$$

We have already seen $o_\diamond(n(P, i)) \in \mathcal{H}_t^\vartheta$ above. Together with $o_\diamond(n(P, i)) < o_\diamond(P)$ and $t \in \mathcal{H}_t^\vartheta$ this implies

$$o_\diamond(n(\mathcal{C}_t P, i)) = \vartheta(t + \Omega^{\circ \diamond}(n(P, i))) \in \mathcal{H}_{t + \Omega^{\circ \diamond}(P)}^\vartheta \subseteq \mathcal{H}_{\mathcal{C}_t P}^\vartheta[[i]].$$

This completes the verification of (H3) for $\mathcal{C}_t P$.

Case $r_\diamond(P) = (\vee_i, \psi_0, \psi_1)$: Set $r_\diamond(\mathcal{C}_t P) = (\vee_i, \psi_0, \psi_1)$ and $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. Local correctness is verified as in the previous case.

Case $r_\diamond(P) = (\forall_x, \psi)$: By assumption (i) the outer quantifier of $\forall_x \psi$ must be bounded, i.e. we must have $\psi \equiv x \notin b \vee \theta$ for some $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formula θ . We set

$$r_\diamond(\mathcal{C}_t P) = (\forall_x, \psi),$$

$$n(\mathcal{C}_t P, a) = \begin{cases} \mathcal{C}_t n(P, a) & \text{if } |a| \leq |b|, \\ \bigvee_{a \notin b, \theta(a)}^0 \text{Ax}_{a \notin b} & \text{if } |a| > |b|. \end{cases}$$

The code $\bigvee_{a \notin b, \theta(a)}^0 \text{Ax}_{a \notin b}$ has been introduced in the proof of Lemma 5.11. To see that $\text{Ax}_{a \notin b}$ is indeed an $\mathbf{L}_{\omega^\alpha}^u$ -code we must check that $|a| > |b|$ implies $a \notin b$: Assume $a \in b$. By definition of the rank we have $b \in \mathbb{L}_{|b|+1}^u$. As the stages of the constructible hierarchy are transitive we obtain $a \in \mathbb{L}_{|b|+1}^u$. Now $|a| \leq |b|$ follows by the minimality of the rank. Next, let us verify that assumptions (i) and (ii) hold for $n(P, a)$ whenever we have $|a| \leq |b|$: Indeed, assumption (i) and the first part of assumption (ii) hold for any $a \in \mathbf{L}_{\omega^\alpha}^u$, as in the previous cases. As for the last part of (ii), observe that b occurs in $\forall_x \psi \in l_\diamond(P)$, which implies $|b| \leq k_\diamond(P)$. Condition (H1) for P provides $k_\diamond(P) \in \mathcal{H}_P^\vartheta$. Thus we see that $\mathcal{H}_P^\vartheta[|a|] = \mathcal{H}_P^\vartheta$ holds whenever we have $|a| \leq |b|$. Using condition (H3) and assumption (ii) for P we can conclude

$$h_1(n(P, a)) \in \mathcal{H}_P^\vartheta[|a|] = \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta,$$

as required by assumption (ii) for $n(P, a)$. Now let us verify condition (L) for the code $\mathcal{C}_t P$: For $|a| \leq |b|$ we have $o_\diamond(n(\mathcal{C}_t P, a)) < o_\diamond(\mathcal{C}_t P)$ as in the previous cases. By Proposition 2.18 the ordinal $o_\diamond(\mathcal{C}_t P) = \vartheta(t + \Omega^{\circ_\diamond(P)})$ is additively principal and bigger than ω . Thus we even have $o_\diamond(n(\mathcal{C}_t P, a)) + \omega \leq o_\diamond(\mathcal{C}_t P)$, as required by condition (L) at the rule (\forall_x, ψ) . For $|a| > |b|$ we compute

$$o_\diamond(n(\mathcal{C}_t P, a)) + \omega = o_\diamond(\bigvee_{a \notin b, \theta(a)}^0 \text{Ax}_{a \notin b}) + \omega = 1 + \omega = \omega < o_\diamond(\mathcal{C}_t P).$$

Still concerning (L), the formula $\forall_x \psi$ occurs in $l_\diamond(P) = l_\diamond(\mathcal{C}_t P)$. For $|a| \leq |b|$ we get

$$l_\diamond(n(\mathcal{C}_t P, a)) = l_\diamond(n(P, a)) \subseteq l_\diamond(P) \cup \{\psi(a)\} = l_\diamond(\mathcal{C}_t P) \cup \{\psi(a)\}$$

from condition (L) for P , and for $|a| > |b|$ we compute

$$\begin{aligned} l_\diamond(n(\mathcal{C}_t P, a)) &= l_\diamond(\bigvee_{a \notin b, \theta(a)}^0 \text{Ax}_{a \notin b}) = l_\diamond(\text{Ax}_{a \notin b} \setminus \{a \notin b\} \cup \{a \notin b \vee \theta(a)\}) = \\ &= \{a \notin b \vee \theta(a)\} = \{\psi(a)\} \subseteq l_\diamond(\mathcal{C}_t P) \cup \{\psi(a)\}. \end{aligned}$$

Condition (H1) is verified as in the previous cases. The same holds for condition (H2) in case $|a| \leq |b|$. For $|a| > |b|$ we compute

$$h_0(n(\mathcal{C}_t P, a)) = h_0(\bigvee_{a \notin b, \theta(a)}^0 \text{Ax}_{a \notin b}) = \Omega \leq t \leq h_1(\mathcal{C}_t P).$$

Condition (H3) for $|a| \leq |b|$ holds as in the previous cases. For $|a| > |b|$ we have

$$h_1(n(\mathcal{C}_t P, a)) \leq k(a \notin b \vee \theta(a)) \leq \max\{k(\forall_x \psi), |a|\} \leq \max\{k_\diamond(P), |a|\}.$$

By condition (H1) and assumption (ii) for P we get $k_\diamond(P) \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_{\mathcal{C}_t P}^\vartheta[|a|]$. Also, $|a| \in \mathcal{H}_{\mathcal{C}_t P}^\vartheta[|a|]$ holds by the definition of our operators. Finally, we have seen above that $|a| > |b|$ implies $o_\diamond(n(\mathcal{C}_t P, a)) = 1$, and $1 \in \mathcal{H}_{\mathcal{C}_t P}^\vartheta[|a|]$ is automatic. This completes the verification of condition (H3) for $\mathcal{C}_t P$.

Case $r_\diamond(P) = (\exists_x, b, \varphi)$: We set $r_\diamond(\mathcal{C}_t P) = (\exists_x, b, \varphi)$ and $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. Observe $|b| \leq k_\diamond(P)$. By condition (H1) and assumption (ii) for P we get

$$|b| \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta.$$

Using Proposition 2.27 we can deduce

$$|b| < \vartheta(t + \Omega^0) \leq \vartheta(t + \Omega^{\circ_\diamond(P)}) = o_\diamond(\mathcal{C}_t P),$$

as condition (L) requires at the rule (\exists_x, b, φ) . The remaining conditions are verified as in the previous cases.

Case $r_\diamond(P) = (\text{Cut}, \psi)$: Note that assumption (i) and condition (C1) for P guarantee $\text{ht}(\psi) < d(P) \leq 2$. In case $\text{ht}(\psi) = 0$ we set $r_\diamond(\mathcal{C}_t P) = (\text{Cut}, \psi)$ and

$n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. To see that $n(P, i)$ satisfies assumptions (i) and (ii), observe $l_{\langle \rangle}(n(P, i)) \subseteq l_{\langle \rangle}(P) \cup \{\psi, \neg\psi\}$ and recall that formulas of height zero are bounded. Condition (C1) for $\mathcal{C}_t P$ holds by

$$\text{ht}(\psi) = 0 < 1 = d(\mathcal{C}_t P).$$

The other conditions are verified as above. Now assume that we have $\text{ht}(\psi) = 1$. It is easy to see that ψ must be of the form $\exists_x \theta$ or $\forall_x \theta$ with the outer quantifier unbounded and θ a bounded formula. We only consider the case $\psi \equiv \exists_x \theta$, because the case $\psi \equiv \forall_x \theta$ is symmetric. Let us give an informal description of the proof idea first: The premise $n(P, 1)$ of the cut rule may contain the formula $\neg\psi \equiv \forall_x \neg\theta$. As this is not a $\Sigma(\mathbf{L}_{\omega_\alpha}^u)$ -formula condition (i) fails for $n(P, 1)$, and we cannot apply the operation \mathcal{C}_t to $n(P, 1)$. Instead, apply \mathcal{C}_t to the premise $n(P, 0)$, which contains the $\Sigma(\mathbf{L}_{\omega_\alpha}^u)$ -formula ψ . This yields a deduction of ψ with height $\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))}) < \Omega$. By the boundedness lemma we obtain a deduction of $\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$. On the other hand we can apply boundedness to the premise $n(P, 1)$, to get a deduction of $\neg\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$ (it makes no difference whether we negate or relativize first). As $\neg\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$ is a $\Sigma(\mathbf{L}_{\omega_\alpha}^u)$ -formula (in fact a bounded formula) we may now collapse the “bounded premise” $n(P, 1)$. Finally, we apply a cut over the bounded formula $\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$. Formally, if $\psi \equiv \exists_x \theta$ with θ bounded then we set

$$\begin{aligned} r_{\langle \rangle}(\mathcal{C}_t P) &= (\text{Cut}, \psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}), \\ n(\mathcal{C}_t P, a) &= \begin{cases} \mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} \mathcal{C}_t n(P, 0) & \text{if } a = 0, \\ \mathcal{C}_{t + \Omega^{o_{\langle \rangle}(n(P, 0))}} \mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} n(P, 1) & \text{if } a \neq 0. \end{cases} \end{aligned}$$

Let us show that this does not lead out of the “intended cases”: Assumptions (i) and (ii) hold for $n(P, 0)$ because of $l_{\langle \rangle}(n(P, 0)) \subseteq l_{\langle \rangle}(P) \cup \{\psi\}$ and because $\psi \equiv \exists_x \theta$ is a $\Sigma(\mathbf{L}_{\omega_\alpha}^u)$ -formula. It follows that we have $o_{\langle \rangle}(\mathcal{C}_t n(P, 0)) = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})$, which fits with the superscript of the function symbol $\mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$. Concerning the case $a \neq 0$, we must verify that assumptions (i) and (ii) hold for the code $\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} n(P, 1)$, with $t + \Omega^{o_{\langle \rangle}(n(P, 0))}$ at the place of t . Now (i) holds by

$$\begin{aligned} l_{\langle \rangle}(\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} n(P, 1)) &= l_{\langle \rangle}(n(P, 1)) \setminus \{\neg\psi\} \cup \{\neg\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P) \cup \{\neg\psi\}) \setminus \{\neg\psi\} \cup \{\neg\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}\} \subseteq l_{\langle \rangle}(P) \cup \{\neg\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}\}, \end{aligned}$$

where $(\neg\psi)^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} \equiv \neg(\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})})$ is a bounded formula, and by

$$d(\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} n(P, 1)) = d(n(P, 1)) \leq d(P) \leq 2.$$

Concerning assumption (ii), we have

$$h_0(\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} n(P, 1)) = h_0(n(P, 1)) \leq h_0(P) \leq t.$$

Also, condition (H3) and assumption (ii) for P yield

$$o_{\langle \rangle}(n(P, 0)) \in \mathcal{H}_P^\vartheta \subseteq \mathcal{H}_t^\vartheta.$$

Together with $t \in \mathcal{H}_t^\vartheta$ this implies

$$t + \Omega^{o_{\langle \rangle}(n(P, 0))} \in \mathcal{H}_t^\vartheta \subseteq \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(n(P, 0))}}^\vartheta,$$

as required by assumption (ii) with $t + \Omega^{\circ \diamond} (n(P, 0))$ at the place of t . Finally, in view of $k(\psi) \leq k((\text{Cut}, \psi)) \leq k_{\diamond}(P)$ we get

$$\begin{aligned} h_1(\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{n(P, 1)} &= \max\{h_1(n(P, 1)), k(\neg\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})\} \leq \\ &\leq \max\{h_1(n(P, 1)), k_{\diamond}(P), \vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))\}. \end{aligned}$$

By conditions (H1, H3) and assumption (ii) for P we obtain

$$\{h_1(n(P, 1)), k_{\diamond}(P)\} \subseteq \mathcal{H}_P^{\vartheta} \subseteq \mathcal{H}_{t + \Omega^{\circ \diamond} (n(P, 0))}^{\vartheta}.$$

We have already seen $t + \Omega^{\circ \diamond} (n(P, 0)) \in \mathcal{H}_{t + \Omega^{\circ \diamond} (n(P, 0))}^{\vartheta}$, which implies

$$\vartheta(t + \Omega^{\circ \diamond} (n(P, 0))) \in \mathcal{H}_{t + \Omega^{\circ \diamond} (n(P, 0))}^{\vartheta}.$$

This completes the verification of conditions (i) and (ii) for $\mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{n(P, 1)}$ and $t + \Omega^{\circ \diamond} (n(P, 0))$. Thus $n(\mathcal{C}_t P, 1) = \mathcal{C}_{t + \Omega^{\circ \diamond} (n(P, 0))} \mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{n(P, 1)}$ falls under the “intended case”. We can now verify condition (L) for $\mathcal{C}_t P$: Similar to the previous cases we have

$$\begin{aligned} o_{\diamond}(n(\mathcal{C}_t P, 0)) &= o_{\diamond}(\mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{\mathcal{C}_t n(P, 0)} = o_{\diamond}(\mathcal{C}_t n(P, 0)) = \\ &= \vartheta(t + \Omega^{\circ \diamond} (n(P, 0))) < \vartheta(t + \Omega^{\circ \diamond} (P)) = o_{\diamond}(\mathcal{C}_t P). \end{aligned}$$

In view of $o_{\diamond}(n(P, 1)) \in \mathcal{H}_P^{\vartheta} \subseteq \mathcal{H}_{t + \Omega^{\circ \diamond} (n(P, 0))}^{\vartheta}$ we also have

$$\begin{aligned} o_{\diamond}(n(\mathcal{C}_t P, 1)) &= o_{\diamond}(\mathcal{C}_{t + \Omega^{\circ \diamond} (n(P, 0))} \mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{n(P, 1)} = \\ &= \vartheta(t + \Omega^{\circ \diamond} (n(P, 0)) + \Omega^{\circ \diamond} (n(P, 1))) < \vartheta(t + \Omega^{\circ \diamond} (n(P, 0)) + \Omega^{\circ \diamond} (P)) = \\ &= \vartheta(t + \Omega^{\circ \diamond} (P)) = o_{\diamond}(\mathcal{C}_t P). \end{aligned}$$

Concerning the end-sequents we have

$$\begin{aligned} l_{\diamond}(n(\mathcal{C}_t P, 0)) &= l_{\diamond}(n(P, 0)) \setminus \{\psi\} \cup \{\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\} \subseteq \\ &\subseteq (l_{\diamond}(P) \cup \{\psi\}) \setminus \{\psi\} \cup \{\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\} \subseteq l_{\diamond}(\mathcal{C}_t P) \cup \{\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\} \end{aligned}$$

and

$$\begin{aligned} l_{\diamond}(n(\mathcal{C}_t P, 1)) &= l_{\diamond}(n(P, 1)) \setminus \{\neg\psi\} \cup \{\neg\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\} \subseteq \\ &\subseteq (l_{\diamond}(P) \cup \{\neg\psi\}) \setminus \{\neg\psi\} \cup \{\neg\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\} \subseteq l_{\diamond}(\mathcal{C}_t P) \cup \{\neg\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}\}, \end{aligned}$$

as required by condition (L) at the rule (Cut, $\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}$). Coming to condition (C1), as $\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}$ is a bounded formula we have

$$\text{ht}(\psi^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))}) = 0 < 1 = d(\mathcal{C}_t P).$$

Conditions (C2) holds by

$$d(n(\mathcal{C}_t P, 0)) = d(\mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{\mathcal{C}_t n(P, 0)} = d(\mathcal{C}_t n(P, 0)) = 1 = d(\mathcal{C}_t P)$$

and

$$d(n(\mathcal{C}_t P, 1)) = d(\mathcal{C}_{t + \Omega^{\circ \diamond} (n(P, 0))} \mathcal{B}_{\forall, \neg\psi}^{\vartheta(t + \Omega^{\circ \diamond} (n(P, 0)))})_{n(P, 1)} = 1 = d(\mathcal{C}_t P).$$

Concerning condition (H1), due to the new parameter in the cut formula we now have

$$k_{\langle \rangle}(\mathcal{C}_t P) \leq \max\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(P)})^*, \vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})\}.$$

As in the previous cases we have

$$\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(P)})^*\} \subseteq \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}.$$

Above we have seen $o_{\langle \rangle}(n(P,0)) \in \mathcal{H}_t^{\vartheta}$. Together with $o_{\langle \rangle}(n(P,0)) < o_{\langle \rangle}(P)$ and $t \in \mathcal{H}_t^{\vartheta}$ this implies

$$\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))}) \in \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^{\vartheta} = \mathcal{H}_{\mathcal{C}_t P}^{\vartheta},$$

completing the verification of (H1). Condition (H2) holds by

$$\begin{aligned} h_0(n(\mathcal{C}_t P, 0)) &= h_0(\mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} \mathcal{C}_t n(P, 0)) = h_0(\mathcal{C}_t n(P, 0)) = \\ &= t + \Omega^{o_{\langle \rangle}(n(P,0))} < t + \Omega^{o_{\langle \rangle}(P)} = h_0(\mathcal{C}_t P) \end{aligned}$$

and

$$\begin{aligned} h_0(n(\mathcal{C}_t P, 1)) &= h_0(\mathcal{C}_{t + \Omega^{o_{\langle \rangle}(n(P,0))}} \mathcal{B}_{\forall, \neg \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} n(P, 1)) = \\ &= t + \Omega^{o_{\langle \rangle}(n(P,0))} + \Omega^{o_{\langle \rangle}(\mathcal{B}_{\forall, \neg \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} n(P, 1))} = t + \Omega^{o_{\langle \rangle}(n(P,0))} + \Omega^{o_{\langle \rangle}(n(P,1))} < \\ &< t + \Omega^{o_{\langle \rangle}(n(P,0))} + \Omega^{o_{\langle \rangle}(P)} = t + \Omega^{o_{\langle \rangle}(P)} = h_0(\mathcal{C}_t P). \end{aligned}$$

Concerning (H3), in view of $k(\psi) = k((\text{Cut}, \psi)) \leq k_{\langle \rangle}(P)$ we have

$$\begin{aligned} h_1(n(\mathcal{C}_t P, 0)) &= h_1(\mathcal{B}_{\exists, \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} \mathcal{C}_t n(P, 0)) = k(\psi^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})}) \leq \\ &\leq \max\{k(\psi), \vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})\} \leq \max\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})\} \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}. \end{aligned}$$

Also, we have

$$h_1(n(\mathcal{C}_t P, 1)) = h_1(\mathcal{C}_{t + \Omega^{o_{\langle \rangle}(n(P,0))}} \mathcal{B}_{\forall, \neg \psi}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} n(P, 1)) = 0 \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}.$$

Finally, we have seen $o_{\langle \rangle}(n(\mathcal{C}_t P, 0)) = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))}) \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}$ above. Also, in view of $o_{\langle \rangle}(n(P, i)) \in \mathcal{H}_P^{\vartheta} \subseteq \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^{\vartheta}$ we have

$$o_{\langle \rangle}(n(\mathcal{C}_t P, 1)) = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))} + \Omega^{o_{\langle \rangle}(n(P,1))}) \in \mathcal{H}_{t + \Omega^{o_{\langle \rangle}(P)}}^{\vartheta} = \mathcal{H}_{\mathcal{C}_t P}^{\vartheta},$$

completing the verification of (H3).

Case $r_{\langle \rangle}(P) = (\text{Ref}, \exists_z \forall_{x \in a} \exists_{y \in z} \theta)$: Recall that we have only allowed this rule if θ is a disjunction. This has the effect that the outer quantifier of the formula $\exists_y \theta$ must be unbounded. Arguing informally, the premise $n(P, 0)$ deduces the $\Sigma(\mathbf{L}_{\omega}^u)$ -formula $\forall_{x \in a} \exists_y \theta$. By the induction hypothesis we can collapse this preproof to height $\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))}) < \Omega$. Boundedness yields a deduction of $\forall_{x \in a} \exists_{y \in \mathbb{L}^u} \theta$.

We can use $\mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})}^u$ as a witness to introduce the existential quantifier over z . Formally we set

$$r_{\langle \rangle}(\mathcal{C}_t P) = (\exists_z, \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})}^u, \forall_{x \in a} \exists_{y \in z} \theta),$$

$$n(\mathcal{C}_t P, a) = \mathcal{B}_{\exists, \forall_{x \in a} \exists_y \theta}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P,0))})} \mathcal{C}_t n(P, 0).$$

In view of $l_{\langle \rangle}(n(P, 0)) \subseteq l_{\langle \rangle}(P) \cup \{\forall_{x \in a} \exists_y \theta\}$ the code $n(P, 0)$ satisfies assumptions (i) and (ii). It follows that $o_{\langle \rangle}(\mathcal{C}_t n(P, 0)) = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})$ fits with the superscript of the function symbol $\mathcal{B}_{\exists, \forall_{x \in a} \exists_y \theta}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}$. Now let us verify local correctness: As in the previous cases we have $o_{\langle \rangle}(n(\mathcal{C}_t P, 0)) + \omega \leq o_{\langle \rangle}(\mathcal{C}_t P)$. In particular this implies

$$|\mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u| = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))}) = o_{\langle \rangle}(n(\mathcal{C}_t P, 0)) < o_{\langle \rangle}(\mathcal{C}_t P),$$

as condition (L) requires at the rule $(\exists_z, \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u, \forall_{x \in a} \exists_y \theta)$. We also have

$$\begin{aligned} l_{\langle \rangle}(n(\mathcal{C}_t P, 0)) &= l_{\langle \rangle}(n(P, 0)) \setminus \{\forall_{x \in a} \exists_y \theta\} \cup \{\forall_{x \in a} \exists_y \in \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u \theta\} \subseteq \\ &\subseteq (l_{\langle \rangle}(P) \cup \{\forall_{x \in a} \exists_y \theta\}) \setminus \{\forall_{x \in a} \exists_y \theta\} \cup \{\forall_{x \in a} \exists_y \in \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u \theta\} \subseteq \\ &\subseteq l_{\langle \rangle}(P) \cup \{\forall_{x \in a} \exists_y \in \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u \theta\} = l_{\langle \rangle}(\mathcal{C}_t P) \cup \{\forall_{x \in a} \exists_y \in \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u \theta\}. \end{aligned}$$

Condition (C1) does not apply, and (C2) holds by

$$d(n(\mathcal{C}_t P, 0)) = d(\mathcal{B}_{\exists, \forall_{x \in a} \exists_y \theta}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} \mathcal{C}_t n(P, 0)) = d(\mathcal{C}_t n(P, 0)) = 1 = d(\mathcal{C}_t P).$$

As for (H1), due to the new parameter $\mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u$ in the rule we have

$$k_{\langle \rangle}(\mathcal{C}_t P) \leq \max\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(P)})^*, \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})\}.$$

We can deduce $k_{\langle \rangle}(\mathcal{C}_t P) \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}$ as in the previous case. Condition (H2) holds by

$$h_0(n(\mathcal{C}_t P, 0)) = h_0(\mathcal{C}_t n(P, 0)) = t + \Omega^{o_{\langle \rangle}(n(P, 0))} < t + \Omega^{o_{\langle \rangle}(P)} = h_0(\mathcal{C}_t P).$$

Concerning (H3), as $\exists_z \forall_{x \in a} \exists_y \theta$ occurs in $l_{\langle \rangle}(P)$ we have $k(\forall_{x \in a} \exists_y \theta) \leq k_{\langle \rangle}(P)$. We can deduce

$$\begin{aligned} h_1(n(\mathcal{C}_t P, 0)) &= h_1(\mathcal{B}_{\exists, \forall_{x \in a} \exists_y \theta}^{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})} \mathcal{C}_t n(P, 0)) = k(\forall_{x \in a} \exists_y \in \mathbb{L}_{\vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})}^u \theta) \leq \\ &\leq \max\{k(\forall_{x \in a} \exists_y \theta), \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})\} \leq \max\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})\}. \end{aligned}$$

As in the previous cases we see

$$\{k_{\langle \rangle}(P), \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))})\} \subseteq \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}.$$

Thus we get $h_1(n(\mathcal{C}_t P, 0)) \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta}$, and also

$$o_{\langle \rangle}(n(\mathcal{C}_t P, 0)) = \vartheta(t + \Omega^{o_{\langle \rangle}(n(P, 0))}) \in \mathcal{H}_{\mathcal{C}_t P}^{\vartheta},$$

as required by condition (H3).

Case $r_{\langle \rangle}(P) = (\text{Rep}, b)$: We set $r_{\langle \rangle}(\mathcal{C}_t P) = (\text{Rep}, b)$ and $n(\mathcal{C}_t P, a) = \mathcal{C}_t n(P, a)$. Note that we have $|b| \leq k_{\langle \rangle}(P)$. Thus condition (H1) for P implies $|b| \in \mathcal{H}_P^{\vartheta}$, which is equivalent to $\mathcal{H}_P^{\vartheta}[|b|] = \mathcal{H}_P^{\vartheta}$. By condition (H3) for P we get

$$o_{\langle \rangle}(n(P, b)) \in \mathcal{H}_P^{\vartheta}[|b|] = \mathcal{H}_P^{\vartheta} \subseteq \mathcal{H}_t^{\vartheta}.$$

With this in mind we can check local correctness as in the previous cases. \square

Putting pieces together we obtain the first theorem from the introduction:

Theorem 5.13. *Working in primitive recursive set theory, consider a countable transitive set u . If the implication*

$$\text{WOP}(\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)) \rightarrow \exists_\alpha \exists_\vartheta \vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$$

holds then there is an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$.

Proof. Fix an enumeration $u = \{u_i \mid i \in \omega\}$. Aiming at a contradiction, assume that there is no admissible set \mathbb{A} with $u \subseteq \mathbb{A}$. By Proposition 2.14 this makes $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ a well-ordering principle. Then the assumption yields a Bachmann-Howard collapse $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$, for some ordinal α . Now consider the $\mathbf{L}_{\omega^\alpha}^u$ -code $\mathcal{C}_\Omega \mathcal{E}^C P_\alpha^u \langle \rangle$, where $C \in \omega$ is as in Proposition 4.14. Let us check that $\mathcal{E}^C P_\alpha^u \langle \rangle$ satisfies assumptions (i) and (ii) of Theorem 5.12, with $t = \Omega$: Proposition 4.14 tells us that the sequent $l_\langle \rangle(\mathcal{E}^C P_\alpha^u \langle \rangle)$ is empty, so trivially it contains only $\Sigma(\mathbf{L}_{\omega^\alpha}^u)$ -formulas. The same proposition ensures $d(\mathcal{E}^C P_\alpha^u \langle \rangle) = 2$, as required by assumption (i) of Theorem 5.12. As for assumption (ii), Definition 5.6 and Lemma 5.9 give $h_0(\mathcal{E}^C P_\alpha^u \langle \rangle) = h_0(P_\alpha^u \langle \rangle) = \Omega$ and $h_1(\mathcal{E}^C P_\alpha^u \langle \rangle) = h_1(P_\alpha^u \langle \rangle) = |\langle \rangle| = 0$. Together with $\Omega \in \mathcal{H}_\Omega^\vartheta$ this establishes assumption (ii) of Theorem 5.12. The theorem thus yields

$$\begin{aligned} l_\langle \rangle(\mathcal{C}_\Omega \mathcal{E}^C P_\alpha^u \langle \rangle) &= \langle \rangle, \\ o_\langle \rangle(\mathcal{C}_\Omega \mathcal{E}^C P_\alpha^u \langle \rangle) &= \vartheta(\Omega + \Omega^{\varepsilon_\langle \rangle}). \end{aligned}$$

By Corollary 4.7 this means that $[\mathcal{C}_\Omega \mathcal{E}^C P_\alpha^u \langle \rangle]$ is an $\mathbf{L}_{\omega^\alpha}^u$ -preproof with empty end-sequent and height $\vartheta(\Omega + \Omega^{\varepsilon_\langle \rangle}) < \Omega$. However, Proposition 3.4 tells us that such a proof cannot exist. Thus we have reached the desired contradiction. \square

It is easy to deduce the implication “(ii) \Rightarrow (iii)” in the second theorem of the introduction: To establish (iii), consider an arbitrary set v . By the axiom of countability there is a countable transitive set u with $v \in u$. Part (ii) provides the implication

$$\text{WOP}(\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)) \rightarrow \exists_\alpha \exists_\vartheta \vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha.$$

Thus the previous theorem yields an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$. In particular we have $v \in \mathbb{A}$, as desired.

6. A WELL-ORDERING PROOF

In the previous sections we have shown that the existence of a Bachmann-Howard collapse $\vartheta : \varepsilon(S_{\omega^\alpha}^u) \xrightarrow{\text{BH}} \alpha$ for one particular well-ordering principle $\alpha \mapsto \varepsilon(S_{\omega^\alpha}^u)$ entails the existence of an admissible set \mathbb{A} with $u \subseteq \mathbb{A}$. The goal of the present section is to establish a converse: Consider some well-ordering principle $T^u : \alpha \mapsto T_\alpha^u$ with rank function $s \mapsto |s|_T^u$. Assume that \mathbb{A} is an admissible set with $u \in \mathbb{A}$, and write

$$o(\mathbb{A}) := \min\{\alpha \in \text{Ord} \mid \alpha \notin \mathbb{A}\} = \text{Ord} \cap \mathbb{A}.$$

We will prove that there is a Bachmann-Howard collapse $\vartheta_{\mathbb{A}} : T_{o(\mathbb{A})}^u \rightarrow o(\mathbb{A})$. The idea is similar to [Rat92a, Section 4]: Perform the construction from Remark 2.6 inside \mathbb{A} , with $o(\mathbb{A})$ at the place of \aleph_1 and “element of \mathbb{A} ” at the place of “countable”. This will lead to a Σ -formula $D_T(u, s, \alpha)$ such that we can set

$$\vartheta_{\mathbb{A}}(s) = \alpha \quad :\Leftrightarrow \quad \mathbb{A} \models D_T(u, s, \alpha)$$

for all $s \in T_{o(\mathbb{A})}^u$ and $\alpha < o(\mathbb{A})$. As a preparation we need to recover primitive recursive functions, in particular the function $\alpha \mapsto T_\alpha^u$, inside \mathbb{A} :

Lemma 6.1. *For each primitive recursive function F there is a Σ -formula φ_F (in the language of pure set theory, i.e. without primitive recursive function symbols) such that the following is provable in primitive recursive set theory: For any*

admissible set \mathbb{A} and any $\vec{x}, y \in \mathbb{A}$ we have

$$F(\vec{x}) = y \quad \Leftrightarrow \quad \mathbb{A} \models \varphi_F(\vec{x}, y),$$

and indeed $F(\vec{x}) \in \mathbb{A}$.

Proof. We argue by (meta-) induction on the build up of primitive recursive set functions (see e.g. [Rat92b, Definition 2.1]). The basic functions and composition are easily accomodated. To prepare definitions by recursion, consider the notion of transitive closure: Let $\text{TC}(\cdot)$ be the canonical primitive recursive function that computes transitive closures. By the proof of [Bar75, Theorem I.6.1] there is a Σ -formula φ_{TC} such that primitive recursive set theory proves

$$\forall_{x,y} (\text{TC}(x) = y \leftrightarrow \varphi_{\text{TC}}(x, y)),$$

as well as

$$\mathbb{A} \models \forall_x \exists_y \varphi_{\text{TC}}(x, y)$$

for any admissible set \mathbb{A} . By upward absoluteness of Σ -formulas $\mathbb{A} \models \varphi_{\text{TC}}(x, y)$ implies $\text{TC}(x) = y$. It follows that \mathbb{A} is closed under transitive closures, and that we have

$$\text{TC}(x) = y \quad \Leftrightarrow \quad \mathbb{A} \models \varphi_{\text{TC}}(x, y).$$

In particular φ_{TC} is a Δ -formula from the viewpoint of \mathbb{A} , namely

$$\mathbb{A} \models \varphi_{\text{TC}}(x, y) \leftrightarrow \forall_{y'} (y' \neq y \rightarrow \neg \varphi_{\text{TC}}(x, y')).$$

Misusing notation we will use the function symbol $\text{TC}(\cdot)$ in formulas of pure set theory: For example $\mathbb{A} \models z \in \text{TC}(x)$ abbreviates either the Σ -formula

$$\mathbb{A} \models \exists_y (\varphi_{\text{TC}}(x, y) \wedge z \in y)$$

or the Π -formula

$$\mathbb{A} \models \forall_y (\varphi_{\text{TC}}(x, y) \rightarrow z \in y),$$

depending on the context. After this preparation, consider a function

$$F(z, \vec{x}) = H(\bigcup \{F(w, \vec{x}) \mid w \in z\}, z, \vec{x})$$

defined by recursion. We repeat the usual proof of Σ -recursion (see [Bar75, Theorem I.6.4]) inside \mathbb{A} : Let φ_H be the Σ -definition of H provided by the induction hypothesis. Set $\varphi_F(z, \vec{x}, y) := \exists_f \theta(z, \vec{x}, y, f)$ where

$$\begin{aligned} \theta(z, \vec{x}, y, f) &\equiv \text{“}f \text{ is a function with domain } \text{TC}(z)\text{”} \wedge \\ &\wedge \forall_{w \in \text{TC}(z)} \varphi_H(\bigcup \text{rng}(f \upharpoonright_w), w, \vec{x}, f(w)) \wedge \varphi_H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}, y). \end{aligned}$$

The claims of the lemma are established by induction on $\text{TC}(z)$ (see [Bar75, Theorem I.6.3]; note that our induction statement is primitive recursive): Concerning “ \Leftarrow ”, assume that we have $\mathbb{A} \models \varphi_F(z, \vec{x}, y)$, i.e. $\mathbb{A} \models \theta(z, \vec{x}, y, f)$ for some $f \in \mathbb{A}$. By the definition of θ we get $\mathbb{A} \models \theta(w, \vec{x}, f(w), f \upharpoonright_{\text{TC}(w)})$ for all $w \in z$. This implies $\mathbb{A} \models \varphi_F(w, \vec{x}, f(w))$, so that the induction hypothesis yields $F(w, \vec{x}) = f(w)$. Furthermore, from $\mathbb{A} \models \theta(z, \vec{x}, y, f)$ we get $\mathbb{A} \models \varphi_H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}, y)$. Using the claim for H we obtain

$$y = H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}) = H(\bigcup \{F(w, \vec{x}) \mid w \in z\}, z, \vec{x}) = F(z, \vec{x}),$$

as required for direction “ \Leftarrow ” of the lemma. As for direction “ \Rightarrow ”, the induction hypothesis gives $F(w, \vec{x}) \in \mathbb{A}$ and $\mathbb{A} \models \varphi_F(w, \vec{x}, F(w, \vec{x}))$ for all $w \in \text{TC}(z)$. Since direction “ \Leftarrow ” provides unicity we obtain

$$\mathbb{A} \models \forall_{w \in \text{TC}(z)} \exists! y \varphi_F(w, \vec{x}, y).$$

Now Σ -replacement (see [Bar75, Theorem I.4.6]) inside \mathbb{A} yields a function $f \in \mathbb{A}$ with domain $\text{TC}(z)$ and

$$\mathbb{A} \models \forall_{w \in \text{TC}(z)} \varphi_F(w, \vec{x}, f(w)).$$

Direction “ \Leftarrow ” tells us $f = F(\cdot, \vec{x}) \upharpoonright_{\text{TC}(z)}$. By the claim for H the value $F(z, \vec{x}) = H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x})$ lies in \mathbb{A} . It remains to show

$$\mathbb{A} \models \theta(z, \vec{x}, H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}), f).$$

The first conjunct of $\theta(z, \vec{x}, H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}), f)$ is immediate. The third conjunct, i.e. the statement

$$\mathbb{A} \models \varphi_H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x}, H(\bigcup \text{rng}(f \upharpoonright_z), z, \vec{x})),$$

holds by the claim for H . Similarly, the second conjunct

$$\mathbb{A} \models \forall_{w \in \text{TC}(z)} \varphi_H(\bigcup \text{rng}(f \upharpoonright_w), w, \vec{x}, f(w))$$

reduces to

$$H(\bigcup \text{rng}(f \upharpoonright_w), w, \vec{x}) = f(w).$$

This is the same as

$$H(\bigcup \{F(u, \vec{x}) \mid u \in w\}, w, \vec{x}) = F(w, \vec{x}),$$

which is the defining clause for F . □

In formulas of pure set theory (which are not supposed to contain symbols for primitive recursive functions) we will use $F(\vec{x}) = y$ as an abbreviation for $\varphi_F(\vec{x}, y)$. As we have seen in the case of transitive closure, the lemma implies that φ_F is a Δ -formula from the viewpoint of any admissible set. We will also use functional notation, e.g. writing $z \in F(\vec{x})$ for either of the formulas $\exists y (y = F(\vec{x}) \wedge z \in y)$ or $\forall y (y = F(\vec{x}) \rightarrow z \in y)$, which are equivalent in any admissible set. The formula $D_T(u, \alpha, s)$ that we have described above will be constructed via the second recursion theorem, applied inside our admissible set:

Lemma 6.2. *Let $C(x_1, \dots, x_n, \vec{y}, R)$ be a Σ -formula involving an n -ary relation symbol R which only occurs positively. Then there is a Σ -formula $D(x_1, \dots, x_n, \vec{y})$ such that we have*

$$\mathbb{A} \models \forall_{x_1, \dots, x_n, \vec{y}} (D(x_1, \dots, x_n, \vec{y}) \leftrightarrow C(x_1, \dots, x_n, \vec{y}, \{x_1, \dots, x_n \mid D(x_1, \dots, x_n, \vec{y})\}))$$

for any admissible set \mathbb{A} .

Proof. This is the second recursion theorem (see [Bar75, Theorem V.2.3]). In the meta-theory we need to construct certain proofs in Kripke-Platek set theory. This is done by induction on formulas, and clearly feasible in primitive recursive set theory (and in much weaker theories). □

In the following we assume that $\alpha \mapsto T_\alpha^u$ is a well-ordering principle with rank function $s \mapsto |s|_T^u$. In particular this means that these functions are primitive recursive. Recall that $T^u = \bigcup_{\alpha \in \text{Ord}} T_\alpha^u$ is a primitive recursive class (namely $s \in T^u$ precisely if $s \in T_{|s|_T^u+1}^u$). We observe that

$$\mathbb{A} \cap T^u = T_{o(\mathbb{A})}^u$$

holds for any admissible set \mathbb{A} : If we have $s \in \mathbb{A} \cap T^u$ then the rank $|s|_T^u$ of s lies in \mathbb{A} as well, by the closure of admissible sets under primitive recursive functions. By the properties of the rank we have $s \in T_{|s|_T^u+1}^u \subseteq T_{o(\mathbb{A})}^u$. On the other hand, $s \in T_{o(\mathbb{A})}^u$ implies $s \in T_\alpha^u$ for some $\alpha \in \mathbb{A}$. Again by closure under primitive recursive functions we get $T_\alpha^u \in \mathbb{A}$. As \mathbb{A} is transitive this implies $s \in T_\alpha^u \subseteq \mathbb{A}$. Using the lemma we can construct a Σ -formula $D_T(u, s, \alpha)$ such that we have

$$\begin{aligned} \mathbb{A} \models D_T(u, s, \alpha) \leftrightarrow & s \in T^u \wedge \alpha \in \text{Ord} \wedge \\ & \exists_a ("a : \omega \rightarrow \text{Ord} \text{ is a function}" \wedge \\ & a(0) = |s|_T^u + 1 \wedge \\ & \forall_{n \in \omega} \exists_d ("d : \{t \in T_{a(n)}^u \mid t <_{T^u} s\} \rightarrow \text{Ord} \text{ is a function}" \wedge \\ & \quad \forall_{t \in \text{dom}(d)} D_T(u, t, d(t)) \wedge \\ & \quad a(n+1) = \sup\{d(t) + 1 \mid t \in \text{dom}(d)\}) \wedge \\ & \alpha = \sup_{n \in \omega} a(n)) \end{aligned}$$

for any admissible set \mathbb{A} with $u \in \mathbb{A}$. Let us begin with uniqueness:

Lemma 6.3. *Let $\mathbb{A} \ni u$ be an admissible set. Then we have*

$$\mathbb{A} \models D_T(u, s, \alpha_0) \wedge D_T(u, s, \alpha_1) \rightarrow \alpha_0 = \alpha_1$$

for all $s \in T_{o(\mathbb{A})}^u$.

Proof. We argue by induction on s (i.e. induction over the well-ordering $<_{T_{o(\mathbb{A})}^u}$). Assume that $\mathbb{A} \models D_T(u, s, \alpha_0)$ and $\mathbb{A} \models D_T(u, s, \alpha_1)$ are witnessed by functions $a_0, a_1 : \omega \rightarrow \text{Ord}$ with $\alpha_i = \sup_{n \in \omega} a_i(n)$. To establish the claim we show $a_0(n) = a_1(n)$ by induction on n . The base $n = 0$ is immediate. Concerning the step, we have

$$a_i(n+1) = \sup\{d_i(t) + 1 \mid t \in \text{dom}(d_i)\}$$

for some functions $d_i : \{t \in T_{a_i(n)}^u \mid t <_{T^u} s\} \rightarrow \text{Ord}$ which satisfy

$$\mathbb{A} \models D_T(u, t, d_i(t))$$

for all $t \in \text{dom}(d_i)$. By induction hypothesis we have $a_0(n) = a_1(n)$, so that the domains of d_0 and d_1 are equal. As $t \in \text{dom}(d_i)$ implies $t <_{T_{o(\mathbb{A})}^u} s$ the main induction hypothesis yields $d_0(t) = d_1(t)$ for all such t . This clearly implies $a_0(n+1) = a_1(n+1)$, as required. \square

After uniqueness we establish existence:

Proposition 6.4. *Let $\mathbb{A} \ni u$ be an admissible set. For all $s \in T_{o(\mathbb{A})}^u$ there is an ordinal $\alpha < o(\mathbb{A})$ with $\mathbb{A} \models D_T(u, s, \alpha)$.*

Proof. Again we argue by induction on s . To establish the claim for s we construct functions $a_m : m+1 \rightarrow \text{Ord}$ in \mathbb{A} such that we have $a_m(0) = |s|_T^u + 1$ and

$$\mathbb{A} \models \exists_d ("d : \{t \in T_{a_m(n)}^u \mid t <_{T^u} s\} \rightarrow \text{Ord is a function}" \wedge$$

$$\forall_{t \in \text{dom}(d)} D_T(u, t, d(t)) \wedge a_m(n+1) = \sup\{d(t) + 1 \mid t \in \text{dom}(d)\})$$

for all $n < m$. The function a_0 is simply the pair $\langle 0, |s|_T^u + 1 \rangle$. To extend a_m to a_{m+1} it suffices to construct a value $a_{m+1}(m+1)$ which satisfies the above condition. First, observe that the set $T_{a_m(m)}^u$ lies in \mathbb{A} by closure under primitive recursive functions. Using Δ -separation in \mathbb{A} we see that $\{t \in T_{a_m(m)}^u \mid t <_{T^u} s\}$ is an element of \mathbb{A} . By the induction hypothesis and the previous lemma we get

$$\mathbb{A} \models \forall_{t \in \{t \in T_{a_m(m)}^u \mid t <_{T^u} s\}} \exists!_{\alpha} D_T(u, t, \alpha).$$

Then Σ -replacement (see [Bar75, Theorem 4.6]) in the admissible set \mathbb{A} provides a function $d : \{t \in T_{a_m(m)}^u \mid t <_{T^u} s\} \rightarrow \text{Ord}$ in \mathbb{A} such that we have $\mathbb{A} \models D_T(u, t, d(t))$ for all $t \in \text{dom}(d)$. Setting

$$a_{m+1} := a_m \cup \{\langle m+1, \sup\{d(t) + 1 \mid t \in \text{dom}(d)\} \rangle\}$$

completes the induction step. Using the previous lemma one checks that the functions $a_m \in \mathbb{A}$ are unique. Thus, by Σ -replacement, the function $m \mapsto a_m$ lies itself in \mathbb{A} . Finally, it follows that the admissible set \mathbb{A} contains the function $a : \omega \rightarrow \text{Ord}$ defined by $a(n) := a_n(n)$. This function witnesses

$$\mathbb{A} \models D_T(u, s, \sup_{n \in \omega} a(n)),$$

which establishes the claim for s . \square

The previous two results justify the following:

Definition 6.5. Consider a well-ordering principle $\alpha \mapsto T_\alpha^u$ and an admissible set \mathbb{A} with $u \in \mathbb{A}$. We define a function $\vartheta_{\mathbb{A}} : T_{o(\mathbb{A})}^u \rightarrow o(\mathbb{A})$ by setting

$$\vartheta_{\mathbb{A}} = \{\langle s, \alpha \rangle \in T_{o(\mathbb{A})}^u \times o(\mathbb{A}) \mid \mathbb{A} \models D_T(u, s, \alpha)\}.$$

Finally, we verify that we have indeed constructed a Bachmann-Howard collapse:

Theorem 6.6. Assume that $\alpha \mapsto T_\alpha^u$ is a well-ordering principle, and that \mathbb{A} is an admissible set with $u \in \mathbb{A}$. Then the function $\vartheta_{\mathbb{A}} : T_{o(\mathbb{A})}^u \rightarrow o(\mathbb{A})$ is a Bachmann-Howard collapse.

Proof. We must verify the two conditions from Definition 2.2. Condition (i) requests $|s|_T^u < \vartheta_{\mathbb{A}}(s)$ for all $s \in T_{o(\mathbb{A})}^u$. To see that this is satisfied, let $a : \omega \rightarrow \text{Ord}$ be a function which witnesses $\mathbb{A} \models D_T(u, s, \vartheta_{\mathbb{A}}(s))$. Then we have

$$|s|_T^u < |s|_T^u + 1 = a(0) \leq \sup_{n \in \omega} a(n) = \vartheta_{\mathbb{A}}(s).$$

Condition (ii) asks us to deduce $\vartheta_{\mathbb{A}}(s) < \vartheta_{\mathbb{A}}(t)$ from $s <_{T_{o(\mathbb{A})}^u} t$ and $|s|_T^u < \vartheta_{\mathbb{A}}(t)$. Let $a : \omega \rightarrow \text{Ord}$ be a witness for $\mathbb{A} \models D_T(u, t, \vartheta_{\mathbb{A}}(t))$. In particular we have $\vartheta_{\mathbb{A}}(t) = \sup_{n \in \omega} a(n)$, and thus $|s|_T^u < a(n)$ for some $n \in \omega$. So we see

$$s \in \{r \in T_{a(n)}^u \mid r <_{T^u} t\}.$$

The definition of $D_T(u, t, \vartheta_{\mathbb{A}}(t))$ yields $a(n+1) = \sup\{d(r) + 1 \mid r \in \text{dom}(d)\}$ for some function $d : \{r \in T_{a(n)}^u \mid r <_{T^u} t\} \rightarrow \text{Ord}$ which satisfies $\mathbb{A} \models D_T(u, r, d(r))$ for all $r \in \text{dom}(d)$. The latter means that d coincides with $\vartheta_{\mathbb{A}}$, so that we get

$$\vartheta_{\mathbb{A}}(s) = d(s) < a(n+1) \leq \sup_{n \in \omega} a(n) = \vartheta_{\mathbb{A}}(t),$$

as required. \square

This establishes “(iii) \Rightarrow (ii)” of the theorem in the introduction.

REFERENCES

- [AR09] Bahareh Afshari and Michael Rathjen. Reverse mathematics and well-ordering principles: A pilot study. *Annals of Pure and Applied Logic*, 160:231–237, 2009.
- [Bar75] Jon Barwise. *Admissible Sets and Structures*, volume 7 of *Perspectives in Mathematical Logic*. Springer, Berlin, 1975.
- [Buc88] Wilfried Buchholz. Induktive Definitionen und Dilatoren. *Archive for Mathematical Logic*, 27:51–60, 1988.
- [Buc91] Wilfried Buchholz. Notation systems for infinitary derivations. *Archive for Mathematical Logic*, 30:277–296, 1991.
- [Buc93] Wilfried Buchholz. A simplified version of local predicativity. In Peter Aczel, Harold Simmons, and Stanley S. Wainer, editors, *Proof Theory: A Selection of Papers From the Leeds Proof Theory Programme 1990*, pages 115–147. Cambridge University Press, 1993.
- [Buc01] Wilfried Buchholz. Finitary treatment of operator controlled derivations. *Mathematical Logic Quarterly*, 3:363–396, 2001.
- [FMW] Harvey Friedman, Antonio Montalbán, and Andreas Weiermann. A characterization of \mathbf{ATR}_0 in terms of a Kruskal-like tree theorem. Unpublished.
- [Gir81] Jean-Yves Girard. Π_2^1 -logic, part 1: Dilators. *Annals of Pure and Applied Logic*, 21:75–219, 1981.
- [Gir85] Jean-Yves Girard. Introduction to Π_2^1 -logic. *Synthese*, 62(2):191–216, 1985.
- [Gir87] Jean-Yves Girard. *Proof Theory and Logical Complexity, Volume 1*. Studies in Proof Theory. Bibliopolis, Napoli, 1987.
- [Hir94] Jeffrey L. Hirst. Reverse mathematics and ordinal exponentiation. *Annals of Pure and Applied Logic*, 66(1):1–18, 1994.
- [Jäg82] Gerhard Jäger. Zur Beweistheorie der Kripke-Platek-Mengenlehre über den natürlichen Zahlen. *Archiv für mathematische Logik und Grundlagenforschung*, 22:121–139, 1982.
- [Jäg86] Gerhard Jäger. *Theories for Admissible Sets. A Unifying Approach to Proof Theory*. Studies in Proof Theory. Bibliopolis, Napoli, 1986.
- [JS99] Gerhard Jäger and Thomas Strahm. Bar induction and ω model reflection. *Annals of Pure and Applied Logic*, 97:221–230, 1999.
- [MM11] Alberto Marcone and Antonio Montalbán. The Veblen functions for computability theorists. *Journal of Symbolic Logic*, 76:575–602, 2011.
- [Mon11] Antonio Montalbán. Open questions in reverse mathematics. *Bulletin of Symbolic Logic*, 17:431–454, 2011.
- [Poh81] Wolfram Pohlers. Cut-elimination for impredicative infinitary systems. Part I. Ordinal analysis for \mathbf{ID}_1 . *Archiv für mathematische Logik und Grundlagenforschung*, 21:113–129, 1981.
- [Poh98] Wolfram Pohlers. Subsystems of set theory and second order number theory. In S. Buss, editor, *Handbook of Proof Theory*, pages 209–335. Elsevier, 1998.
- [Rat92a] Michael Rathjen. Fragments of Kripke-Platek set theory. In Peter Aczel, Harold Simmons, and Stanley S. Wainer, editors, *Proof Theory*, pages 251–273. Cambridge University Press, 1992.
- [Rat92b] Michael Rathjen. A proof-theoretic characterization of the primitive recursive set functions. *Journal of Symbolic Logic*, 57(3):954–969, 1992.
- [Rat11] Michael Rathjen. Omega models and well-ordering principles, beta models and functors. Talk at the Reverse Mathematics Workshop, University of Chicago, 17 September 2011.
- [Rat14] Michael Rathjen. ω -models and well-ordering principles. In Neil Tennant, editor, *Foundational Adventures: Essays in Honor of Harvey M. Friedman*, pages 179–212. College Publications, 2014.
- [RV15] Michael Rathjen and Pedro Francisco Valencia Vizcaíno. Well ordering principles and bar induction. In Reinhard Kahle and Michael Rathjen, editors, *Gentzen’s centenary: The quest for consistency*, pages 533–561. Springer, Berlin, 2015.
- [RW11] Michael Rathjen and Andreas Weiermann. Reverse mathematics and well-ordering principles. In S. Barry Cooper and Andrea Sorbi, editors, *Computability in Context: Computation and Logic in the Real World*, pages 351–370. Imperial College Press, 2011.
- [Sch77] Kurt Schütte. *Proof Theory*, volume 225 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, Heidelberg and New York, 1977.

- [Sim09] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Logic. Cambridge University Press, 2009.